On Characterizations of Real Hypersurfaces in a Complex Space Form with n-Parallel Shape Operator

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On characterizations of real hypersurfaces in a complex space form with η-parallel shape operator

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Abstract

In this paper, we study real hypersurfaces in a non-flat complex space form with η-parallel shape operator. Several partial characterizations of these real hypersurfaces are obtained.

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1 Introduction

Let $M_n(c)$ be an n-dimensional complete and simply connected non-flat complex space form with constant holomorphic sectional curvature $4c$, i.e., it is either a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$ (according to as the holomorphic sectional curvature $4c$ is positive or negative). Suppose M is a connected real hypersurface in $M_n(c)$ and N is a unit normal vector field of M. Then the complex structure J of $M_n(c)$ induces an almost contact metric structure $(\phi, \xi, \eta, \langle , \rangle)$ on M, i.e.,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = \langle \xi, X \rangle.$$

We denote by $\Gamma(\mathcal{V})$ the module of all differentiable sections on the vector bundle $\mathcal{V}$ over M. Typical examples of real hypersurfaces are the homogeneous real hypersurfaces M. In 1973, Takagi [17] classified these homogeneous real hypersurfaces in $\mathbb{C}P^n$ into six types, so-called real hypersurfaces of type A_1, A_2, B, C, D and E. A *Hopf hypersurface* M in $M_n(c)$ is characterized by the condition that the structure vector field ξ is principal, i.e., $A\xi = \alpha\xi$, and it can be shown that this principal curvature α is a constant.

By looking at Takagi's classification, one may verify that the homogeneous real hypersurfaces are Hopf and with constant principal curvatures. In 1986, Kimura [7] showed that the converse is also true, i.e., Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}P^n$ are in fact those real hypersurfaces of type A_1, A_2, etc. Also, Berndt [2] showed a $\mathbb{C}H^n$'s version for Kimura's result, i.e., Hopf hypersurfaces with constant principal curvatures could be divided into four types, nowadays known as type A_0, A_1, A_2 and B. In what follows, by real hypersurfaces of type A, we mean of type A_1, A_2 (resp. of type A_0, A_1, A_2) for $c > 0$ (resp. for $c < 0$). Other than these Hopf hypersurfaces, another example of real hypersurfaces in $M_n(c)$ are the class of ruled real hypersurfaces. *Ruled real hypersurfaces* in $M_n(c)$ are characterized by having a one-codimensional foliation whose leaves are complex totally geodesic hyperplanes in $M_n(c)$. The geometry of ruled real hypersurfaces in $M_n(c)$ was studied in [10].

One of the first result in the theory of real hypersurfaces M in $M_n(c)$ is the shape operator A of M in $M_n(c)$ cannot be parallel, i.e., $\nabla A \neq 0$, where ∇ is the Levi-Civita connection of M. The non-existence of real hypersurfaces in $M_n(c)$ with parallel shape operator motivates the study of the weaker notion of η-parallelism, which was first introduced by Kimura and Maeda [8]. The shape operator A is said to be *η-parallel* if it satisfies the following condition:

$$\langle(\nabla_X A)Y, Z\rangle = 0$$

for any X, Y and $Z \in \Gamma(\mathcal{D})$, where $\mathcal{D} := \operatorname{span}\{\xi\}^{\perp}$, called the *holomorphic distribution* on M. The complete classification of real hypersurfaces with η-parallel shape operator in $M_n(c)$ remain open up to this point, nevertheless, many partial characterizations have been obtained either by imposing an additional condition or by considering a condition that is slightly stronger than the η-parallelism (for instance, cf.[1, 4, 5, 8, 15, 16], etc). It is worthy to note that real hypersurfaces that appeared in the list of these characterizations are those of type A, B and ruled real hypersurfaces.

In this paper, we shall continue the study of real hypersurfaces in $M_n(c)$ with η-parallel shape operator. In particular, several partial characterizations of real hypersurfaces in $M_n(c)$ with η-parallel shape operator are obtained.

This paper is organized as follows. Section 2 recalls some basic formulas and briefly reviews certain known results on real hypersurfaces in $M_n(c)$ with η-parallel shape operator. Some auxiliary lemmas are derived in Section 3. In Section 4 we focus on contact real hypersurfaces in $M_n(c)$ and give a characterization for ruled real hypersurfaces and contact real hypersurfaces. In Section 5 we characterize real hypersurfaces in $M_n(c)$ with η-parallel shape operator under the commutativity assumption on $\phi A \phi$ and $\phi^2 A \phi^2$. In the last section we characterize real hypersurfaces in $M_n(c)$ with prescribed covariant derivative of the shape operator.

2 Preliminaries

Consider a connected real hypersurface M in $M_n(c)$, the induced almost contact metric structure $(\phi, \xi, \eta, \langle,\rangle)$ on M has the following properties

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1 \tag{1}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y\rangle\xi, \quad \nabla_X \xi = \phi AX \tag{2}$$

for any $X, Y \in \Gamma(TM)$. Let R be the curvature tensor of M. Then the equations of Gauss and Codazzi are given respectively by

$$R(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle \phi Y,Z\rangle \phi X - \langle \phi X,Z\rangle \phi Y \\ -2\langle \phi X,Y\rangle \phi Z\} + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY$$

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X,Y\rangle \xi\}.$$

The second order covariant derivative $\nabla_X\nabla_Y A$ on the shape operator A is defined by

$$(\nabla_X\nabla_Y A)Z = \nabla_X\{(\nabla_Y A)Z\} - (\nabla_{\nabla_X Y}A)Z - (\nabla_Y A)\nabla_X Z.$$

Next, we state two necessary and sufficient conditions for real hypersurfaces in $M_n(c)$ to be of type A.

Theorem 2.1 ([3, 11, 12, 14]). *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Then the following are equivalent:*

1. *M is locally congruent to one of real hypersurfaces of type A;*
2. *$\phi A = A\phi$;*
3. *$(\nabla_X A)Y = -c\{\langle \phi X,Y\rangle \xi + \eta(Y)\phi X\}$, for any $X, Y \in \Gamma(TM)$.*

The following theorem, proved by Kimura and Maeda, and Suh respectively for $c > 0$ and $c < 0$, completely classified Hopf hypersurfaces with η-parallel shape operator in $M_n(c)$.

Theorem 2.2 ([8, 16]). *Let M be a Hopf hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel shape operator. Then M is locally congruent to one of real hypersurfaces of type A and B.*

The above theorem is not true if the condition that M being Hopf is removed.

Theorem 2.3 ([1, 8]). *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies the following two conditions:*

1. *$\phi(\phi A + A\phi)\phi = 0$, i.e., the holomorphic distribution $\mathcal{D}$ is integrable;*
2. *the shape operator A is η-parallel.*

Then M is locally congruent to a ruled real hypersurface.

On the other hand, Ki and Suh studied real hypersurfaces M with η-parallel shape operator without assuming it is Hopf. By restricting Condition 2 and Condition 3 in Theorem 2.1 to the holomorphic distribution $\mathcal{D}$, they obtained the following result.

Theorem 2.4 ([4]). *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies the following two conditions:*

1. *$\phi(\phi A - A\phi)\phi = 0$,*
2. *$(\nabla_X A)Y = -c\langle \phi X,Y\rangle \xi$, for any $X, Y \in \Gamma(\mathcal{D})$.*

Then M is locally congruent to one of real hypersurfaces of type A.

Observe that Condition 2 in this theorem is a special form for the shape operator A to be η-parallel. Ahn, Lee and Suh weaken it to the η-parallelism condition on A and proved the following.

Theorem 2.5 ([1]). *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies the following two conditions:*

1. $\phi(\phi A - A\phi)\phi = 0$,
2. *the shape operator A is η-parallel.*

Then M is locally congruent to a ruled real hypersurface or one of real hypersurfaces of type A and B.

The above theorem gave a significant improvement of Theorem 2.4 as it allows all the standard examples of real hypersurfaces with η-parallel shape operator to be included in the list of characterization.

Before we end this section, we shall state the expression of ∇A on these standard examples of real hypersurfaces with η-parallel shape operator.

Theorem 2.6. *Let M be real hypersurface in $M_n(c)$,$n \geq 3$, and $X, Y \in \Gamma(\mathcal{D})$.*

1. *If M is of type A then*
$$(\nabla_X A)Y = -c\langle \phi X, Y\rangle \xi.$$
2. *If M is of type B then*
$$(\nabla_X A)Y = \{-c\langle \phi X, Y\rangle + \frac{\alpha}{2}\langle (\phi A - A\phi)X, Y\rangle\}\xi.$$
3. *If M is ruled and $V = \phi A\xi$ then*
$$(\nabla_X A)Y = \{-c\langle \phi X, Y\rangle + \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle\}\xi.$$

Statement 1 above is an immediate consequence of Statement 3 in Theorem 2.1 while Statement 3 above was derived in [15]. In order to verify Statement 2, we need to recall a lemma.

Lemma 2.7 ([6]). *Let M be a real hypersurfaces in $M_n(c)$. If $A\xi = \alpha\xi$ then α is a constant and $(\nabla_\xi A) = (\alpha/2)(\phi A - A\phi)$.*

Since the shape operator of real hypersurfaces of type B is η-parallel, for $X, Y \in \Gamma(\mathcal{D})$ Statement 2 in the above theorem can be derived as follows:

$$\begin{aligned}(\nabla_X A)Y &= \langle (\nabla_X A)Y, \xi\rangle \xi \\ &= \{-\langle \phi X, Y\rangle + \langle (\nabla_\xi A)X, Y\rangle\}\xi \quad \text{(by the Codazzi equation)} \\ &= \{-c\langle \phi X, Y\rangle + \frac{\alpha}{2}\langle (\phi A - A\phi)X, Y\rangle\}\xi \quad \text{(by Lemma 2.7)}.\end{aligned}$$

3 Real hypersurfaces with non-principal structure vector field

Hopf hypersurfaces with η-parallel shape operator A have already been completely characterized in Theorem 2.2. In this section, we focus on real hypersurfaces M on which the structure vector field ξ is not principal, or equivalently, with the restriction $\beta := ||\phi A\xi|| \neq 0$. Certain auxiliary lemmas that are needed in the following sections are also derived here.

We shall first fix some notations as follows: $V := \nabla_\xi \xi = \phi A\xi$, $\alpha := \eta(A\xi)$ and $F := \nabla_\xi A$. Then it is clear that the shape operator A of a real hypersurface M is η-parallel if and only if

$$(\nabla_X A)Y = \{-c\langle\phi X, Y\rangle + \langle FX, Y\rangle\}\xi, \quad X, Y \in \Gamma(\mathcal{D}).$$

The next lemma plays an important role in the rest of the paper.

Lemma 3.1. *Let M be a real hypersurface in $M_n(c)$ with η-parallel shape operator A. Then*

$$\begin{aligned}
&c\{\langle Y, AZ\rangle\langle X, W\rangle - \langle X, AZ\rangle\langle Y, W\rangle \\
&+ \langle\phi Y, AZ\rangle\langle\phi X, W\rangle - \langle\phi X, AZ\rangle\langle\phi Y, W\rangle - 2\langle\phi X, Y,\rangle\langle\phi AZ, W\rangle \\
&- \langle Y, Z\rangle\langle X, AW\rangle + \langle X, Z\rangle\langle Y, AW\rangle \\
&- \langle\phi Y, Z\rangle\langle\phi X, AW\rangle + \langle\phi X, Z\rangle\langle\phi Y, AW\rangle + 2\langle\phi X, Y,\rangle\langle\phi Z, AW\rangle\} \\
&+ \langle AY, AZ\rangle\langle AX, W\rangle - \langle AX, AZ\rangle\langle AY, W\rangle \\
&- \langle AY, Z\rangle\langle AX, AW\rangle + \langle AX, Z\rangle\langle AY, AW\rangle \\
=&c\{\langle Z, \phi AY\rangle\langle\phi X, W\rangle + \langle W, \phi AY\rangle\langle\phi X, Z\rangle \\
&- \langle Z, \phi AX\rangle\langle\phi Y, W\rangle - \langle W, \phi AX\rangle\langle\phi Y, Z\rangle\} \\
&+ \langle Y, \phi AX\rangle\langle FZ, W\rangle + \langle Z, \phi AX\rangle\langle FY, W\rangle + \langle W, \phi AX\rangle\langle FZ, Y\rangle \\
&- \langle X, \phi AY\rangle\langle FZ, W\rangle - \langle Z, \phi AY\rangle\langle FX, W\rangle - \langle W, \phi AY\rangle\langle FZ, X\rangle
\end{aligned}$$

for any $X, Y, Z, W \in \Gamma(\mathcal{D})$.

Proof. For any $Y, Z, W \in \Gamma(\mathcal{D})$, by differentiating the following equation covariantly

$$\langle(\nabla_Y A)Z, W\rangle = 0$$

in the direction of $X \in \Gamma(\mathcal{D})$, we obtain

$$\langle(\nabla_X\nabla_Y A)Z + (\nabla_{\nabla_X Y}A)Z + (\nabla_Y A)\nabla_X Z, W\rangle + \langle(\nabla_Y A)Z, \nabla_X W\rangle = 0.$$

From the η-parallellism condition and (2), the above equation reduces to

$$\begin{aligned}\langle(\nabla_X\nabla_Y A)Z, W\rangle = \langle Y, \phi AX\rangle\langle(\nabla_\xi A)Z, W\rangle + \langle Z, \phi AX\rangle\langle(\nabla_Y A)\xi, W\rangle \\ +\langle W, \phi AX\rangle\langle(\nabla_Y A)Z, \xi\rangle.\end{aligned}$$

Furthermore, by using the Codazzi equation, the above equation becomes

$$\begin{aligned}\langle(\nabla_X\nabla_Y A)Z, W\rangle = \langle Y, \phi AX\rangle\langle FZ, W\rangle + \langle Z, \phi AX\rangle\{\langle FY, W\rangle - c\langle\phi Y, W\rangle\}\\ +\langle W, \phi AX\rangle\{\langle FY, Z\rangle - c\langle\phi Y, Z\rangle\}.\end{aligned}$$

Finally, by the Ricci identity $(R(X,Y)A)Z = (\nabla_X\nabla_Y A)Z - (\nabla_Y\nabla_X A)Z$ and the above equation, we obtain the lemma. □

Lemma 3.2. *Let M be a real hypersurface in $M_n(c)$ with η-parallel shape operator A. Then*

$$-\langle A\phi V, Y\rangle\langle\phi V, X\rangle + \langle A\phi V, X\rangle\langle\phi V, Y\rangle = \langle\frac{\tau}{2}(\phi A + A\phi)X + (F\phi A + A\phi F)X, Y\rangle$$

for any $X, Y \in \Gamma(\mathcal{D})$, where $\tau := -\operatorname{trace}\phi F\phi$.

Proof. Let $E_1, E_2, \cdots, E_{2n-2}$ be a local field of orthonormal frames in $\Gamma(\mathcal{D})$. By putting $Z = W = E_j$, for $j = 1, 2, \cdots, 2n-2$, in Lemma 3.1 and then summing up these equations, we get

$$\begin{aligned}2\langle\phi A^2Y, \phi AX\rangle - 2\langle\phi A^2X, \phi AY\rangle = \sum_{j=1}^{2n-2}\langle FE_j, E_j\rangle\langle\phi AX + A\phi X, Y\rangle\\ +2\langle FY, \phi AX\rangle - 2\langle FX, \phi AY\rangle.\end{aligned}$$

Next, by applying (1) in the left hand side of this equation, we obtain the lemma. □

Lemma 3.3. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel shape operator A. Suppose that β is nowhere zero on M. If there exist two functions ν and $\tilde{\nu}$ such that*

$$AV = \nu V \quad \text{and} \quad A\phi V = \tilde{\nu}\phi V - \beta^2\xi$$

then $\phi A\phi$ and $\phi^2 A\phi^2$ can be diagonalized simultaneously.

Proof. Let x be an arbitrary point in M. From the hypothesis, the subspace

$$\mathcal{H} := \operatorname{span}\{V, \phi V, \xi\}$$

and its orthogonal complement $\mathcal{H}^\perp$ in T_xM are both invariant by A and hence by both $\phi A\phi$ and $\phi^2 A\phi^2$ as well. Furthermore, each eigenvector $E \in \mathcal{H}^\perp$ of $\phi^2 A\phi^2$ is a principal vector as well. If ϕE is principal, for each principal vector $E \in \mathcal{H}^\perp$ then the statement is clearly true. Hence, we suppose that there is a unit principal vector $E' \in \mathcal{H}^\perp$ but $\phi E'$ is not principal.

Firstly, by letting $X, Y \in \mathcal{H}^\perp$, $Z = V$ and $W = \phi V$ in Lemma 3.1, we obtain

$$-2c\beta^2(\nu - \tilde{\nu})\langle\phi X, Y\rangle = \langle F\phi V, V\rangle\langle(\phi A + A\phi)X, Y\rangle. \tag{3}$$

Since $\phi E'$ is not principal, we can see that $\langle F\phi V, V\rangle = 0 = \nu - \tilde{\nu}$, (for otherwise, by putting $X = E'$ in the above equation, yields $A\phi E' = \tilde{\lambda}\phi E'$ and a contradiction).

Next, by putting $X = \phi V$, $Y = V$ in Lemma 3.1 and making use of the fact that $\nu = \tilde{\nu}$,

$$\begin{aligned} &2c\beta^2\langle(\phi A - A\phi)Z, W\rangle - \nu\beta^2\{\langle V, Z\rangle\langle\phi V, W\rangle + \langle\phi V, Z\rangle\langle V, W\rangle\} \\ &= -\,2\nu\beta^2\langle FZ, W\rangle - \nu\{\langle V, Z\rangle\langle FV, W\rangle + \langle V, W\rangle\langle FV, Z\rangle \\ &\quad + \langle\phi V, Z\rangle\langle F\phi V, W\rangle + \langle\phi V, W\rangle\langle F\phi V, Z\rangle\}. \qquad (4) \end{aligned}$$

If we put $Z, W \in \mathcal{H}^{\perp}$ in (4), then

$$c\langle(\phi A - A\phi)Z, W\rangle = -\nu\langle FZ, W\rangle. \qquad (5)$$

From the hypothesis $\phi E'$ is not principal, the right hand side of (5) is not identically zero, so we may assume that $\nu \neq 0$. On the other hand, by putting $Z = V$ and $W = \phi V$ in (4), and taking account of $\langle FV, \phi V\rangle = \nu - \tilde{\nu} = 0$, we obtain $-\nu\beta^6 = 0$. This contradicts the facts $\nu \neq 0$ and $\beta \neq 0$. The proof is completed. □

4 Characterizations on contact real hypersurfaces

An almost contact manifold $(M^{2n-1}, \phi, \xi, \eta)$ is said to be a *contact manifold* if

$$\eta \wedge (d\eta)^{n-1} \neq 0$$

on M. If there is a Riemannian metric $\langle , \rangle$ which is compatible with this contact structure then $(\phi, \xi, \eta, \langle , \rangle)$ becomes a *contact metric structure* and M is said to be a *contact metric manifold.*

A real hypersurface in a Kaehler manifold is said to be *contact* if its induced almost contact structure is contact. Okumura proved a necessary and sufficient condition for real hypersurfaces in a Kaehler manifold to be contact.

Theorem 4.1 ([13]). *Let M be a real hypersurface in a Kaehler manifold. Then the induced almost contact structure (ϕ, ξ, η) is contact if and only if there is a non-vanishing function k on M such that*

$$\phi A + A\phi - k\phi = 0. \qquad (6)$$

It can be shown that k is constant. Kon proved the following characterization while the ambient space is $\mathbb{C}P^n$.

Theorem 4.2 ([9]). *Let M be a complete real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. If M satisfies*

$$\phi A + A\phi - \varepsilon\phi = 0$$

for some nonzero constant ε, then M is congruent to one of real hypersurface of type A_1 and B.

On the other hand, Vernon gave a characterization of contact real hypersurfaces in $\mathbb{C}H^n$.

Theorem 4.3 ([18]). *Let M be a complete contact real hypersurface in $\mathbb{C}H^n$, $n \geq 3$. Then M is congruent to one of real hypersurface of type A_0, A_1 and B.*

In this section, we study real hypersurfaces in $M_n(c)$ under a weaker version of (6), i.e.,

$$\phi(\phi A + A\phi - k\phi)\phi = 0, \tag{7}$$

for some function k on M. We shall first derive some identities from the condition (7). Note that (7) is equivalent to

$$\langle(\phi A + A\phi - k\phi)Y, Z\rangle = 0, \quad Y, Z \in \Gamma(\mathcal{D}).$$

Differentiating this equation covariantly in the direction of $X \in \Gamma(\mathcal{D})$ we get

$$\begin{aligned}\langle\phi AY, \nabla_X Z\rangle + \langle(\nabla_X\phi)AY + \phi(\nabla_X A)Y + \phi A\nabla_X Y, Z\rangle \\ +\langle A\phi Y, \nabla_X Z\rangle + \langle(\nabla_X A)\phi Y + A(\nabla_X\phi)Y + A\phi\nabla_X Y, Z\rangle \\ -(Xk)\langle\phi Y, Z\rangle - k\langle\phi Y, \nabla_X Z\rangle - k\langle(\nabla_X\phi)Y + \phi\nabla_X Y, Z\rangle = 0.\end{aligned}$$

By using (2) and (7), this equation can be reformed as

$$\begin{aligned}-\langle Z, V\rangle\langle\phi AX, Y\rangle + \langle Y, V\rangle\langle\phi AX, Z\rangle - \langle(\nabla_X A)Y, \phi Z\rangle + \langle(\nabla_X A)Z, \phi Y\rangle \\ +\eta(AY)\langle AX, Z\rangle - \eta(AZ)\langle AX, Y\rangle - (Xk)\langle\phi Y, Z\rangle = 0.\end{aligned}$$

Now by replacing X, Y and Z cyclically in the above equation and then summing these equations, with the help of the Codazzi equation and (7), we obtain

$$\mathfrak{S}(k\langle X, V\rangle + Xk)\langle\phi Y, Z\rangle = 0$$

where $\mathfrak{S}$ denotes the cyclic sum over X, Y and Z. Let X be an arbitrary vector field in $\Gamma(\mathcal{D})$. If we choose $Y \perp X, \phi X$ and $Z = \phi Y$ in the above equation then $k\langle X, V\rangle + Xk = 0$.

We summarize the above observations in the following lemma.

Lemma 4.4. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies*

$$\phi(\phi A + A\phi - k\phi)\phi = 0$$

for some function k on M. Then for any X, Y and $Z \in \Gamma(\mathcal{D})$,

$$\begin{aligned}-\langle Z, V\rangle\langle\phi AX, Y\rangle + \langle Y, V\rangle\langle\phi AX, Z\rangle - \langle(\nabla_X A)Y, \phi Z\rangle + \langle(\nabla_X A)Z, \phi Y\rangle \\ +\eta(AY)\langle AX, Z\rangle - \eta(AZ)\langle AX, Y\rangle - (Xk)\langle\phi Y, Z\rangle = 0,\end{aligned} \tag{8}$$

$$k\langle X, V\rangle + Xk = 0. \tag{9}$$

We first look at the case where k is a nonzero constant. In this case, the equation (9) implies that $V = 0$, i.e., ξ is principal and so $(\phi A + A\phi - k\phi)\xi = 0$. Consequently, we have $\phi A + A\phi - k\phi = 0$, for some nonzero constant k, and hence it follows from Theorem 4.2 and Theorem 4.3 that we obtain

Theorem 4.5. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. If M satisfies*

$$\phi(\phi A + A\phi - \varepsilon\phi)\phi = 0$$

for some constant $\varepsilon \neq 0$, then M is locally congruent to one of real hypersurface of type A_0, A_1 and B.

On the other hand, by adding the η-parallelism condition on the shape operator, we have the following characterization.

Theorem 4.6. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies the following two conditions:*

(i) $\phi(\phi A + A\phi - k\phi)\phi = 0$, for some function k on M;

(ii) the shape operator A is η-parallel.

Then M is locally congruent to a ruled real hypersurface or one of real hypersurface of type A_0, A_1 and B.

Proof. In this case, the equation (8) can be reduced as

$$\begin{aligned} -\langle Z, V\rangle\langle \phi AX, Y\rangle + \langle Y, V\rangle\langle \phi AX, Z\rangle \\ +\eta(AY)\langle AX, Z\rangle - \eta(AZ)\langle AX, Y\rangle - (Xk)\langle \phi Y, Z\rangle = 0. \end{aligned}$$

If we choose $Y \perp V, \phi V$ and $Z = \phi Y$ then $Xk = 0$, for all $X \in \Gamma(\mathcal{D})$ and together with (9), we obtain $k\langle V, V\rangle = 0$ on M. Since we are studying local geometry, we may assume that either $k = 0$ on M or k is nowhere zero on M. If k is identically zero then M is ruled by Theorem 2.3. If k is nowhere zero on M, ξ is principal and so $(\phi A + A\phi - k\phi)\xi = 0$. Consequently, we have $\phi A + A\phi - k\phi = 0$, and hence our result follows from Theorem 4.2 and Theorem 4.3. □

5 Real hypersurfaces with a commutative condition

Observe that the Condition 1 in Theorem 2.3, Theorem 2.5 and Theorem 4.6 imply that $\phi^2 A\phi^2$ and $\phi A\phi$ are commutative. Hence, it is natural to ask if the Condition 1 in these theorems is replaceable by this condition. The main purpose of this section is to give an affirmative answer to this question. We first prove the following lemma.

Lemma 5.1. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel shape operator A. If $\phi A\phi$ and $\phi^2 A\phi^2$ commute then either*

(i) $\phi(\phi A - A\phi)\phi = 0$, or

(ii) $\phi(\phi A + A\phi - k\phi)\phi = 0$ for some function k on M.

Proof. As $\phi A\phi$ and $\phi^2 A\phi^2$ are commutative, they can be diagonalized simultaneously and hence there is a local field of orthonormal frames E_j, ϕE_j $(1 \leq j \leq n-1)$ on $\Gamma(\mathcal{D})$ such that

$$\begin{aligned} AE_j =& e_j\xi + \lambda_j E_j \\ A\phi E_j =& \tilde{e}_j\xi + \tilde{\lambda}_j \phi E_j. \end{aligned}$$

By making the following substitutions for the vectors X, Y, Z and W in Lemma 3.1:

(a) $Y = Z = E_i$, $W = X = \phi E_j$, $(i \neq j)$;

(b) $Y = Z = E_i$, $W = X = E_j$, $(i \neq j)$;

(c) $Y = Z = E_i$, $W = X = \phi E_i$;

(d) $X = E_j$, $Y = \phi E_j$, $Z = \phi E_i$, $W = E_i$, $(i \neq j)$,

we obtain the following equations

$$\tilde{\lambda}_j\lambda_i^2 - (\tilde{\lambda}_j^2 - c + \tilde{e}_j^2)\lambda_i + (e_i^2 - c)\tilde{\lambda}_j = 0 \tag{10}$$

$$\lambda_j\lambda_i^2 - (\lambda_j^2 - c + e_j^2)\lambda_i + (e_i^2 - c)\lambda_j = 0 \tag{11}$$

$$(\lambda_i - \tilde{\lambda}_i)(\lambda_i\tilde{\lambda}_i + 5c) + \tilde{\lambda}_i e_i^2 - \lambda_i\tilde{e}_i^2 + 2\langle FE_i, \phi E_i\rangle(\lambda_i + \tilde{\lambda}_i) = 0 \tag{12}$$

$$2c(\lambda_i - \tilde{\lambda}_i) + (\lambda_j + \tilde{\lambda}_j)\langle FE_i, \phi E_i\rangle = 0. \tag{13}$$

If $\lambda_i = \tilde{\lambda}_i$ for all i then $\phi(\phi A - A\phi)\phi = 0$ and we obtain Statement (i). Hence, we suppose $\lambda_i \neq \tilde{\lambda}_i$ for some i, says $\lambda_1 \neq \tilde{\lambda}_1$. From (13), we obtain $\langle FE_1, \phi E_1\rangle \neq 0$ and

$$\lambda_r + \tilde{\lambda}_r = 2c\frac{\tilde{\lambda}_1 - \lambda_1}{\langle FE_1, \phi E_1\rangle}, \quad r \neq 1. \tag{14}$$

We consider two cases: (I) $\lambda_s \neq \tilde{\lambda}_s$ for some $s \neq 1$; and (II) $\lambda_r = \tilde{\lambda}_r$ for all $r \neq 1$.

Case (I) $\lambda_s \neq \tilde{\lambda}_s$ for some $s \neq 1$, says $\lambda_2 \neq \tilde{\lambda}_2$.

From (13), we obtain $\langle FE_2, \phi E_2\rangle \neq 0$ and

$$\lambda_s + \tilde{\lambda}_s = 2c\frac{\tilde{\lambda}_2 - \lambda_2}{\langle FE_2, \phi E_2\rangle}, \quad s \neq 2. \tag{15}$$

By observing (14) and (15), we obtain

$$\lambda_i + \tilde{\lambda}_i = 2c\frac{\tilde{\lambda}_1 - \lambda_1}{\langle FE_1, \phi E_1\rangle}, \quad \text{for all } i. \tag{16}$$

Therefore, we obtain Statement (ii) with $k = 2c(\tilde{\lambda}_1 - \lambda_1)\langle FE_1, \phi E_1\rangle^{-1}$.

Case (II) $\lambda_r = \tilde{\lambda}_r$ for all $r \neq 1$.

In this case, (14) reduces to

$$\lambda_r = \tilde{\lambda}_r = c\frac{\tilde{\lambda}_1 - \lambda_1}{\langle FE_1, \phi E_1\rangle} \neq 0, \quad r \neq 1.$$

On the other hand, taking $j = 1$ and $i \neq 1$, and then by taking the operation $\lambda_j \times (10) - \tilde{\lambda}_j \times (11)$, yields

$$(\lambda_1 - \tilde{\lambda}_1)(\lambda_1\tilde{\lambda}_1 + c) + \tilde{\lambda}_1 e_1^2 - \lambda_1 \tilde{e}_1^2 = 0.$$

From this equation and (12), we can see

$$\lambda_1 + \tilde{\lambda}_1 = 2c\frac{\tilde{\lambda}_1 - \lambda_1}{\langle FE_1, \phi E_1\rangle}.$$

Adding this case into (14), we also obtain (16) and Statement (ii). This completes the proof. □

It follows from Theorem 2.5, Theorem 4.6 and Lemma 5.1 that we have

Theorem 5.2. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$, with η-parallel shape operator A. If $\phi A\phi$ and $\phi^2 A\phi^2$ commute then M is locally congruent to a ruled real hypersurface or one of real hypersurface of type A and B.*

6 Real hypersurfaces with prescribed covariant derivative of the shape operator

In the previous sections, we characterized real hypersurfaces M with η-parallel shape operator A under certain additional conditions on M. In this section we study these real hypersurfaces from another aspect, i.e., by looking at a condition that is slightly stronger than the η-parallelism on A.

In Theorem 2.6 we see that these "standard examples" of real hypersurfaces with η-parallel shape operator have a nice form for the covariant derivative of the shape operator on the holomorphic distribution $\mathcal{D}$. Motivated by these identities, it is natural to ask if the converse of the identities in Theorem 2.1 are true. In 1995, Suh proved the following

Theorem 6.1 ([15]). *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. If M satisfies*

$$(\nabla_X A)Y = \{-c\langle\phi X, Y\rangle + \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle\}\xi$$

for any $X, Y \in \Gamma(\mathcal{D})$, then M is locally congruent to a ruled real hypersurface or a real hypersurface of type A.

It follows from the above theorem that, since $V = 0$ is necessary and sufficient for ξ to be principal, we can easily obtain the following characterization for real hypersurfaces of type A.

Corollary 6.2. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies*

$$(\nabla_X A)Y = -c\langle\phi X, Y\rangle\xi$$

for any $X, Y \in \Gamma(\mathcal{D})$. Then M is locally congruent to a real hypersurface of type A.

The condition in Theorem 6.1 is too strong to be used to characterize all the standard examples of real hypersurfaces with η-parallel shape operator. It shall be replaced by a weaker condition in order to broaden the list of characterization. In this sense, we have the following.

Theorem 6.3. *Let M be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose M satisfies*

$$(\nabla_X A)Y = \{-c\langle\phi X, Y\rangle + \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle \\ +\varepsilon\langle(\phi A - A\phi)X, Y\rangle\}\xi \tag{17}$$

for any $X, Y \in \Gamma(\mathcal{D})$, where ε is a constant. Then M is locally congruent to a ruled real hypersurface or one of real hypersurfaces of type A and B.

Proof. The condition (17) implies that A is η-parallel. If ξ is principal then by virtue of Theorem 2.1 and Theorem 2.2, we conclude that M is of type A or B. Hence, we may suppose that β is nowhere zero on M. On the other hand, with the condition (17), the tensor field F takes the form

$$\langle FX, Y\rangle = \eta(AY)\langle X, V\rangle + \eta(AX)\langle Y, V\rangle + \varepsilon\langle(\phi A - A\phi)X, Y\rangle \tag{18}$$

for any $X, Y \in \Gamma(\mathcal{D})$. It follows from this equation that $\tau = -\operatorname{trace}\phi F\phi = 0$. Moreover, the identity in Lemma 3.2 can be reduced to

$$-\langle AX, V\rangle\langle Y, V\rangle + \langle AY, V\rangle\langle X, V\rangle = \varepsilon\langle(A\phi A\phi - \phi A\phi A)X, Y\rangle. \tag{19}$$

First, by putting $X = V$ and $Y = \phi V$ in (19), we obtain $\langle AV, \phi V\rangle = 0$. Next, if we put $Y = \phi V$ in (19) then

$$\varepsilon\langle(A\phi A + \phi A\phi A\phi)V, X\rangle = 0 \tag{20}$$

for any $X \in \Gamma(\mathcal{D})$. Finally, when we put $Y = V$ in (19), we get

$$\begin{aligned} \beta^2\langle AX, V\rangle - \langle AV, V\rangle\langle X, V\rangle &= \varepsilon\langle\phi X, (A\phi A + \phi A\phi A\phi)V\rangle \\ &= 0 \qquad \text{(from (20))}. \end{aligned}$$

This equation tells us that $AV = \nu V$. Next, we wish to prove that $A\phi V = \tilde{\nu}\phi V - \beta^2\xi$. For this purpose, we put $Y = \phi V$ and $Z = W = V$ in Lemma 3.1, then

$$0 = c\{\beta^2\langle A\phi V, X\rangle - \langle A\phi V, \phi V\rangle\langle\phi V, X\rangle\} + \frac{\langle FV, V\rangle}{2}\langle\phi A\phi V - \nu V, X\rangle \\ -\langle A\phi V, \phi V\rangle\langle FV, X\rangle + \langle FV, \phi V\rangle\langle A\phi V, X\rangle.$$

On the other hand, by putting $Y = V$ and $Z = W = \phi V$ in Lemma 3.1, we get

$$c\{\beta^2\langle A\phi V, \phi X\rangle - \langle A\phi V, \phi V\rangle\langle V, X\rangle\} = \frac{\langle F\phi V, \phi V\rangle}{2}\langle\nu\phi V + A\phi V, X\rangle \\ +\nu\{\beta^2\langle F\phi V, X\rangle - \langle FV, \phi V\rangle\langle V, X\rangle\}.$$

By using (18), the above two equations becomes

$$(\beta^2 - \varepsilon\nu - c)\{\beta^2\langle A\phi V, X\rangle - \langle A\phi V, \phi V\rangle\langle\phi V, X\rangle\} = 0$$
$$(\varepsilon\nu + c)\{\beta^2\langle A\phi V, \phi X\rangle - \langle A\phi V, \phi V\rangle\langle\phi V, \phi X\rangle\} = 0$$

for any $X \in \Gamma(\mathcal{D})$. From these two equations and the fact that $\beta \neq 0$,

$$\langle A\phi V, X\rangle = \beta^{-2}\langle A\phi V, \phi V\rangle\langle\phi V, X\rangle, \quad X \in \Gamma(\mathcal{D})$$

and hence we have $A\phi V = \tilde{\nu}\phi V - \beta^2\xi$, where $\tilde{\nu} = \beta^{-2}\langle A\phi V, \phi V\rangle$. According to Lemma 3.3 and Theorem 5.2, we conclude that M is ruled and this completes the proof. □

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