Preservers of Pairs of Bivectors with Bounded Distance

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ABSTRACT

We extend Liu's fundamental theorem of the geometry of alternate matrices to the second exterior power of an infinite dimensional vector space and also use her theorem to characterize surjective mappings T from the vector space V of all $n \times n$ alternate matrices over a field with at least three elements onto itself such that for any pair A, B in V, $\operatorname{rank}(A - B) \leqslant 2k$ if and only if $\operatorname{rank}(T(A) - T(B)) \leqslant 2k$, where k is a fixed positive integer such that $n \geqslant 2k + 2$ and $k \geqslant 2$.

1. Introduction

Let F be a field. Let $M_{m,n}(F)$ denote the vector space of all $m \times n$ matrices over F. Let ρ denote the rank function. Two matrices $A, B \in M_{m,n}(F)$ are called adjacent if $\rho(A - B) = 1$. Hua [4] proved the following fundamental theorem of the geometry of rectangular matrices. If ϕ is a bijective mapping on $M_{m,n}(F)$, $m, n \geqslant 2$, $|F| > 2$, such that ϕ preservers adjacency in both directions, then there exist $R \in M_{m,n}(F)$, an $m \times m$ invertible matrix P, an $n \times n$ invertible matrix Q, and an automorphism σ of F such that one of the following holds:

(i) $\phi(A) = PA_\sigma Q + R, A \in M_{m,n}(F)$; or
(ii) $m = n, \phi(A) = PA_\sigma^t Q + R, A \in M_{m,n}(F)$.

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Here A_σ is the matrix obtained from A by applying σ entrywise, $A_\sigma = (\sigma(a_{ij}))$.

By using Hua's theorem, Havlicek and Šemrl [2] characterized bijective mappings ϕ on $M_{m,n}(F)$, $m, n \geqslant 2$, $|F| > 2$, such that for every pair $A, B \in M_{m,n}(F)$, $A - B$ is of full rank if and only if $\phi(A) - \phi(B)$ is of full rank. Motivated by their work, Lim and Tan [6] characterized surjective mappings T from $M_{m,n}(F)$ onto itself such that $\rho(A - B) \leqslant k$ if and only if $\rho(T(A) - T(B)) \leqslant k$, where $|F| > 2$ and k is a fixed positive integer $< \min\{m, n\}$.

Let $K_n(F)$ denote the vector space of all $n \times n$ alternate matrices over F. Liu [7] proved the following fundamental theorem of the geometry of alternate matrices. If $\phi : K_n(F) \to K_n(F)$, $n \geqslant 5$, is a bijective mapping such that

$$\rho(A - B) = 2 \text{ if and only if } \rho(\phi(A) - \phi(B)) = 2,$$

then there exist an $n \times n$ invertible matrix P, an $n \times n$ alternate matrix R, an automorphism σ on F and a nonzero scalar λ such that

$$T(A) = \lambda P A_\sigma P^t + R$$

for any A in $K_n(F)$.

In this note, we first extend Liu's fundamental theorem of the geometry of alternate matrices to the second exterior power of an infinite dimensional vector space. We next use Liu's theorem to characterize surjective mappings ϕ from $K_n(F)$ onto itself such that for any pair A, B in $K_n(F)$, $\rho(A - B) \leqslant 2k$ if and only if $\rho(\phi(A) - \phi(B)) \leqslant 2k$ where $|F| \geqslant 3$ and k is a fixed positive integer such that $2k + 2 \leqslant n$. In the last section, we use this characterization theorem and a result of Wan [12] to describe surjective mappings ϕ from the space of all $n \times n$ symmetric matrices onto itself such that $\rho(A - B) \leqslant 2k$ if and only if $\rho(\phi(A) - \phi(B)) \leqslant 2k$ where k is a fixed positive integer such that $n \geqslant 2k + 1 \geqslant 5$ and F is a perfect field of characteristic two with $|F| > 3$.

2. Adjacency preserving mappings on second exterior powers

Throughout this paper, F is a field, U and W are vector spaces over F of dimension at least two. Let $\Lambda^2 U$ be the second exterior power of U. Let k be a positive integer. An element A of $\Lambda^2 U$ is said to have length k if k is the smallest positive integer such that A is the sum of k nonzero decomposable elements. It is an elementary fact that $A \in \Lambda^2 U$ is of length k if and only if

$$A = \sum_{i=1}^{k} u_{2i-1} \wedge u_{2i}$$

for some linearly independent vectors $u_1, \ldots, u_{2k}$ in U. In this case, we write $l(A) = k$. As usual, the zero element in $\Lambda^2 U$ is said to have length zero.

Let A and B be two elements in $\Lambda^2 U$. We call $l(A - B)$ the arithmetic distance between A and B, and we say that A, B are adjacent if $l(A - B) = 1$.

If $A \in \Lambda^2 U$ is of length $n > 0$ and

$$A = \sum_{i=1}^{n} x_{2i-1} \wedge x_{2i} = \sum_{i=1}^{n} u_{2i-1} \wedge u_{2i}$$

for some $x_j, u_j \in U, j = 1, \ldots, 2n$, then it is known that

$$\langle x_1, \ldots, x_{2n} \rangle = \langle u_1, \ldots, u_{2n} \rangle,$$

where $\langle x_1, \ldots, x_{2n} \rangle$ is the linear span of the vectors $x_1, \ldots, x_{2n}$ and we shall use $[A]$ to denote this uniquely determined subspace $\langle x_1, \ldots, x_{2n} \rangle$.

Let f be a semilinear mapping from U to W associated with an automorphism σ on F. Let $\Lambda^2 f$ denote the second induced power of f from $\Lambda^2 U$ to $\Lambda^2 W$ where

$$(\Lambda^2 f)(u_1 \wedge u_2) = f(u_1) \wedge f(u_2)$$

for any u_1, u_2 in U.

Let f be a σ-semilinear mapping from U to W and g be a τ-semilinear mapping from U to W. If $\Lambda^2 f = d\Lambda^2 g$ for some scalar d in F and the rank of f is at least three, then it can be shown that $\sigma = \tau$ and $f = \lambda g$ for some nonzero scalar λ in F such that $\lambda^2 = d$. This fact will be used several times in our proof of the following theorem.

Theorem 2.1. *Let U and W be vector spaces over F where U is infinite dimensional. Let $T : \Lambda^2 U \to \Lambda^2 W$ be a surjective mapping that preserves adjacency in both directions. Then there exist an invertible semilinear mapping $\varphi : U \to W$, an element R in $\Lambda^2 W$ and a nonzero scalar λ in F such that*

$$T(A) = \lambda(\Lambda^2\varphi)(A) + R$$

for any A in $\Lambda^2 U$.

Proof. We first show that T is injective. Suppose that $T(A) = T(B)$ and $C := B - A$ is a nonzero element. Define a mapping $\phi : \Lambda^2 U \to \Lambda^2 W$ by $\phi(X) = T(X + A) - T(A)$. Then $\phi(0) = \phi(C) = 0$ and ϕ preserves adjacency in both directions. Since U is infinite dimensional, one can choose a nonzero decomposable element D in $\Lambda^2 U$ such that $[C] \cap [D] = \{0\}$. Clearly C and D are not adjacent. However $\phi(C)$ and $\phi(D)$ are adjacent, a contradiction. Hence $C = 0$ and T is injective.

Let $S : \Lambda^2 U \to \Lambda^2 W$ be the mapping defined by $S(X) = T(X) - T(0)$. Then S is a bijective mapping preserving adjacency in both directions and $S(0) = 0$. Let G and H be two elements in $\Lambda^2 U$ such that $l(G - H) = s > 0$ and $l(S(G) - S(H)) = t$. Since there exists a chain $G_0 = G, G_1, \ldots, G_s = H$ of elements in $\Lambda^2 U$ such that $l(G_{i+1} - G_i) = 1$, $i = 0, 1, \ldots, s-1$ and S preserves adjacency, it follows that $t \leqslant s$. Similarly by considering S^{-1}, we have $s \leqslant t$. Hence $l(S(G) - S(H)) = l(G - H)$.

Let X be any $2m$-dimensional subspace of U spanned by $x_1, \ldots, x_{2m}$ where $m \geqslant 2$. Let

$$S(x_{2i-1} \wedge x_{2i}) = y_{2i-1} \wedge y_{2i}, \quad i = 1, \ldots, m.$$

We shall show that $S(\Lambda^2 X) = \Lambda^2 Y$ where $Y = \langle y_1, \ldots, y_{2m} \rangle$. Let k be the largest integer $\leqslant m$ such that $y_1, \ldots, y_{2k}$ are linearly independent and let $J = \sum_{i=1}^{k} x_{2i-1} \wedge x_{2i}$. Then for each $1 \leqslant i \leqslant k$, $S(J) - y_{2i-1} \wedge y_{2i}$ has length $k - 1$. Hence

$$S(J) = y_{2i-1} \wedge y_{2i} + J_i$$

for some J_i of length $k - 1$. Since $l(S(J)) = k$, it follows that

$$[S(J)] = \langle y_{2i-1}, y_{2i} \rangle + [J_i]$$

and hence

$$\langle y_{2i-1}, y_{2i} \rangle \subseteq [S(J)], \quad i = 1, \ldots, k.$$

This shows that $[S(J)] = \langle y_1, \ldots, y_{2k} \rangle$. Suppose that $k < m$. Then the length of $S(J) - S(x_{2k+1} \wedge x_{2k+2})$ is less than $k + 1$. However, $J - x_{2k+1} \wedge x_{2k+2}$ is of length $k + 1$, a contradiction. Hence $k = m$. This implies that $[S(J)] = Y$.

Let $u \wedge v$ be a nonzero decomposable vector in $\Lambda^2 X$ and $S(u \wedge v) = u' \wedge v'$. Then

$$l(u \wedge v - J) \leqslant m.$$

This implies that

$$l(u' \wedge v' - S(J)) \leqslant m.$$

Hence $\langle u', v' \rangle \cap Y \neq \{0\}$. We shall show that $\langle u', v' \rangle \subseteq Y$. Suppose the contrary. Then $\dim(\langle u', v' \rangle + Y) = 2m + 1$. We may assume that $v' \notin Y$. Choose $w \in W$ such that $w, v', y_1, \ldots, y_{2m}$ are linearly independent. Then

$$l(w \wedge v' - S(J)) = m + 1$$

and hence

$$l(S^{-1}(w \wedge v') - J) = m + 1. \tag{1}$$

Note that $u' \wedge v'$, $w \wedge v'$ are adjacent and hence

$$u \wedge v,\ S^{-1}(w \wedge v') \text{ are adjacent.}$$

Hence $[S^{-1}(w \wedge v')] \cap X \neq \{0\}$, a contradiction to (1). Hence $\langle u', v' \rangle \subseteq Y$. This shows that $S(u \wedge v) \in \Lambda^2 Y$.

Now let E be an element of length t in $\Lambda^2 X$ with $t \geqslant 2$. Then

$$E = \sum_{i=1}^{t} u_{2i-1} \wedge u_{2i}$$

for some linearly independent vectors $u_1, \ldots, u_{2t}$ in X. Let

$$S(u_{2i-1} \wedge u_{2i}) = v_{2i-1} \wedge v_{2i}, \quad i = 1, \ldots, t.$$

Then $\langle v_{2i-1}, v_{2i} \rangle \subseteq Y$. In view of the argument in paragraph 3, we have

$$[S(E)] = \langle v_1, \ldots, v_{2t} \rangle.$$

Hence $S(E) \in \Lambda^2 Y$. This proves that $S(\Lambda^2 X) \subseteq \Lambda^2 Y$. Using S^{-1} and applying the previous arguments, we have $S^{-1}(\Lambda^2 Y) \subseteq \Lambda^2 X$. Thus $S(\Lambda^2 X) = \Lambda^2 Y$.

Let A, B be two elements in $\Lambda^2 U$. Let Q be a $2s$-dimensional subspace of U containing $[A] \cup [B]$ where $s > 2$. Then $S(\Lambda^2 Q) = \Lambda^2 K$ for some subspace K of W with $\dim Q = \dim K$. Then by the fundamental theorem of geometry of alternate matrices, we obtain that $S|_{\Lambda^2 Q}$ is additive. This shows that $S(A + B) = S(A) + S(B)$ and hence S is additive.

We now fix a 6-dimensional subspace Z of U. Then by Liu's theorem, we have

$$S|_{\Lambda^2 Z} = \lambda \Lambda^2 f$$

for some invertible σ-semilinear mapping f from Z onto $f(Z)$ and some nonzero scalar λ in F.

Let N be a basis of a complementary subspace of Z in U. For any $e \in N$, choose $e' \in U$ such that $\langle e, e' \rangle \cap Z = \{0\}$ and let $M = Z + \langle e, e' \rangle$. Let $L = \lambda^{-1} S$. Then by Liu's theorem,

$$L|_{\Lambda^2 M} = \eta \Lambda^2 g$$

for some invertible τ-semilinear mapping g from M to $g(M)$ and some nonzero scalar η. Since $\Lambda^2 f = \eta \Lambda^2 g$ on $\Lambda^2 Z$, we have $\sigma = \tau$ and $f = dg$ on Z for some $d \in F$ such that $\eta = d^2$. Hence $L|_{\Lambda^2 M} = \Lambda^2(dg)$. Extend f to a σ-semilinear mapping f_e on $Z + \langle e \rangle$ by defining $f_e(e) = dg(e)$. Then we have

$$L|_{\Lambda^2(Z + \langle e \rangle)} = \Lambda^2 f_e.$$

Let e_1, e_2 be two distinct elements from N. Then

$$L|_{\Lambda^2(Z + \langle e_1, e_2 \rangle)} = \delta \Lambda^2 h$$

for some invertible σ-semilinear mapping h from $Z + \langle e_1, e_2 \rangle$ to its image and some nonzero scalar δ. Since $L|_{\Lambda^2(Z + \langle e_i \rangle)} = \Lambda^2 f_{e_i}$ for $i = 1, 2$, it follows that there exist scalars d_1, d_2 such that $d_i^2 = \delta$ and

$$f_{e_1}(z) = d_1 h(z) \quad \text{for every } z \in Z + \langle e_1 \rangle$$

and

$$f_{e_2}(z) = d_2 h(z) \quad \text{for every } z \in Z + \langle e_2 \rangle.$$

Hence $d_1 = d_2$. For any $\mu \in F$, we have

$$\begin{aligned} L(\mu e_1 \wedge e_2) &= \delta h(\mu e_1) \wedge h(e_2) \\ &= d_1 h(\mu e_1) \wedge d_2 h(e_2) \\ &= f_{e_1}(\mu e_1) \wedge f_{e_2}(e_2). \end{aligned}$$

Let φ be the additive mapping from U to W such that $\varphi|_{Z+\langle e\rangle} = f_e$ for every $e \in N$. Then φ is an invertible σ-semilinear mapping from $\Lambda^2 U$ to $\Lambda^2 W$. Since L is additive, we have $L|_{\Lambda^2(Z+\langle e_1,e_2\rangle)} = \Lambda^2\varphi$. Since e_1, e_2 are arbitrarily chosen from N, it follows that $L = \Lambda^2\varphi$. This completes the proof. □

Remark 2.2. In [8], Hua's fundamental theorem of the geometry of rectangular matrices was extended to the algebra of bounded linear operators of finite rank on an infinite dimensional Banach space, while in [1], Hua's fundamental theorem of the geometry of complex symmetric matrices [3] was extended to the spaces of all bounded symmetric operators of finite rank on an infinite dimensional complex Hilbert space. We remark that Hua's fundamental theorem of the geometry of rectangular matrices holds true under a weaker assumption of preserving the adjacency in one direction only (see [5,9]).

3. Preservers of pairs of bivectors with bounded distance

Throughout this section, F is a field with at least three elements and k is a fixed positive integer such that $6 \leqslant 2+2k \leqslant \dim U$. For any nonempty subset S of $\Lambda^2 U$, let

$$S^{\perp_k} = \{B \in \Lambda^2 U : l(B-A) \leqslant k \ \forall A \in S\}.$$

For characterizing surjective mappings from $\Lambda^2 U$ to $\Lambda^2 W$ that preserve pairs of bivectors with bounded distance k in both directions, we need the following three lemmas.

Lemma 3.1. *Let $A \in \Lambda^2 U$ be a nonzero element of length $\leqslant k$. If $C \in \{0,A\}^{\perp_k\perp_k}$ and $C \neq 0$, then $[C] = [A]$.*

Proof. Let $l(A) = s$. Then

$$A = \sum_{i=1}^{s} x_{2i-1} \wedge x_{2i}$$

for some linearly independent vectors $x_1, \ldots, x_{2s}$ in U. Let $l(C) = t$.

Case 1: $s > t$. Clearly there exist linearly independent vectors $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$ where $m = 2(s-t)$ and $1 \leqslant j_1 < j_2 < \cdots < j_m \leqslant 2s$ such that

$$\langle x_{j_1}, x_{j_2}, \ldots, x_{j_m}\rangle \cap [C] = \{0\}.$$

Let E be a tensor of length $k+1-s$ in $\Lambda^2 U$ such that

$$[E] \cap (\langle x_{j_1}, x_{j_2}, \ldots, x_{j_m}\rangle + [C]) = \{0\}.$$

Let us choose two of the indices $j_1, j_2, \ldots, j_m$, say j_1, j_2. If $\{j_1, j_2\} = \{2r-1, 2r\}$ for some $1 \leqslant r \leqslant s$, let

$$D = E + \sum_{i=1}^{s-t} x_{j_{2i-1}} \wedge x_{j_{2i}}.$$

Then $l(D) = k-t+1$ and $l(D-A) \leqslant k$. Hence $D \in \{0,A\}^{\perp_k}$. However $l(C-D) = k+1$, a contradiction.

If $\{j_1, j_2\} \neq \{2r-1, 2r\}$ for any $1 \leqslant r \leqslant s$, we may assume without loss of generality that $j_1 = 1, j_2 = 3$. Note that there exists a scalar $\lambda \in F$ such that

$$\lambda x_3 + x_2 \notin [C] + \langle x_1, x_{j_3}, \ldots, x_{j_m}\rangle.$$

Let $D = x_1 \wedge (\lambda x_3 + x_2) + \sum_{i=2}^{s-t} x_{j_{2i-1}} \wedge x_{j_{2i}} + E$. Then $l(D-A) \leqslant k$ and $l(D) \leqslant k$ and hence $D \in \{0,A\}^{\perp_k}$. However $l(D-C) = k+1$, a contradiction.

Case 2: $s < t$. Let $D \in \Lambda^2 U$ be of length $k-t+1$ such that $[D] \cap [C] = \{0\}$. Since $l(D-A) \leqslant (k-t+1)+s \leqslant k$, it follows that $D \in \{0,A\}^{\perp_k}$. However, $l(D-C) = k+1$, a contradiction.

From Case 1 and Case 2, we see that $s = t$. Suppose that $[C] \neq [A]$. Then there exists $x_j \in [A]$ such that $x_j \notin [C]$. Extend x_j to $2(k-t)+2$ linearly independent vectors $x_j, v_0, v_1, \ldots, v_{2k-2t}$ such that

$$\langle x_j, v_0, v_1, \ldots, v_{2k-2t}\rangle \cap [C] = \{0\}.$$

Let $J = x_j \wedge v_0 + v_1 \wedge v_2 + \cdots + v_{2k-2t-1} \wedge v_{2k-2t}$. Then $J \in \{0, A\}^{\perp_k}$. But $l(C - J) = k + 1$, a contradiction. Hence $[C] = [A]$. □

Lemma 3.2. *Let A, B be an adjacent pair in $\Lambda^2 U$. Then $|\{A, B\}^{\perp_k \perp_k}| = |F|$.*

Proof. Without loss of generality, we may assume that $B = 0$ and A is a nonzero decomposable tensor. Let $D \in \{0, A\}^{\perp_k \perp_k}$. Then by Lemma 3.1, $D = \lambda A$ for some scalar λ. Now for any $C \in \{0, A\}^{\perp_k}$, we have $l(C) \leqslant k$ and hence $\dim[C] \leqslant 2k$. Since $l(A - C) \leqslant k$, it follows that

$$\dim([A] + [C]) \leqslant 2k + 1.$$

Hence $l(\mu A - C) \leqslant k$ for any $\mu \in F$. This shows that $\mu A \in \{0, A\}^{\perp_k \perp_k}$ and hence

$$\{0, A\}^{\perp_k \perp_k} = \langle A \rangle.$$

This completes the proof. □

Lemma 3.3. *Let $A, B \in \Lambda^2 U$ such that $2 \leqslant l(A - B) \leqslant k$. Then $\{A, B\}^{\perp_k \perp_k} = \{A, B\}$.*

Proof. Without loss of generality, we may assume that $A = 0$ and

$$B = \sum_{i=1}^{n} x_{2i-1} \wedge x_{2i}$$

for some linearly independent vectors $x_1, \ldots, x_{2n}$. Then $2 \leqslant n \leqslant k$. Choose vectors $x_{2n+1}, \ldots, x_{2k+2}$ in U such that $x_1, \ldots, x_{2n}, \ldots, x_{2k+2}$ are linearly independent.

Let $C \in \{0, B\}^{\perp_k \perp_k}$ and $C \neq 0$. By Lemma 3.1, $[C] = [B]$ and thus

$$C = \sum_{1 \leqslant i < j \leqslant n} a_{ij} x_i \wedge x_j$$

for some $a_{ij} \in F$ such that $a_{ji} = -a_{ij}$.

Let $1 \leqslant s < t \leqslant n$. For each $\lambda(t) = (\lambda_1, \ldots, \hat{\lambda}_t, \ldots, \lambda_k) \in F^{k-1}$, let

$$\begin{aligned} F_{st}(\lambda(t)) = & \sum_{1=r \neq s,t}^{k} \lambda_r x_{2r-1} \wedge x_{2r} + x_{2s-1} \wedge x_{2s} + x_{2t-1} \wedge x_{2t} \\ & + \lambda_s x_{2s-1} \wedge x_{2k+1} + x_{2t-1} \wedge x_{2k+2}. \end{aligned} \tag{2}$$

Then $F_{st}(\lambda(t)) \in \{0, B\}^{\perp_k}$ and hence $l(C - F_{st}(\lambda(t))) \leqslant k$. This implies that

$$\begin{vmatrix} 0 & a_{2s-1,2s} - 1 & a_{2s-1,2t-1} & a_{2s-1,2t} & -\lambda_s & 0 \\ 1 - a_{2s-1,2s} & 0 & a_{2s,2t-1} & a_{2s,2t} & 0 & 0 \\ -a_{2s-1,2t-1} & -a_{2s,2t-1} & 0 & a_{2t-1,2t} - 1 & 0 & -1 \\ -a_{2s-1,2t} & -a_{2s,2t} & 1 - a_{2t-1,2t} & 0 & 0 & 0 \\ \lambda_s & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{vmatrix} = 0.$$

Since the above determinant equals $a_{2s,2t}^2 \lambda_s^2$, it follows that $a_{2s,2t} = 0$.

Replacing $\lambda_s x_{2s-1} \wedge x_{2k+1} + x_{2t-1} \wedge x_{2k+2}$ by

(i) $\lambda_s x_{2s} \wedge x_{2k+1} + x_{2t-1} \wedge x_{2k+2}$;
(ii) $\lambda_s x_{2s-1} \wedge x_{2k+1} + x_{2t} \wedge x_{2k+2}$; and
(iii) $\lambda_s x_{2s} \wedge x_{2k+1} + x_{2t} \wedge x_{2k+2}$

in (2) respectively, we get (i) $a_{2s-1,2t} = 0$; (ii) $a_{2s,2t-1} = 0$; and (iii) $a_{2s-1,2t-1} = 0$ respectively. Hence

$$C = \sum_{i=1}^{n} a_{2i-1,2i} x_{2i-1} \wedge x_{2i}.$$

For each $1 \leqslant s < t \leqslant n$, let $\lambda(s,t) = (\lambda_1, \ldots, \hat{\lambda}_s, \ldots, \hat{\lambda}_t, \ldots, \lambda_{k+1}) \in F^{k-1}$ and

$$G(\lambda(s,t)) = \sum_{1=r\neq s,t}^{k+1} \lambda_r x_{2r-1} \wedge x_{2r} + x_{2s-1} \wedge x_{2s}.$$

Then $G(\lambda(s,t)) \in \{0, B\}^{\perp_k}$ and hence

$$l(C - G(\lambda(s,t))) \leqslant k$$

and we get $(a_{2s-1.2s} - 1)^2 a_{2t-1,2t}^2 = 0$. Since $l(C) = n$, we have $a_{2t-1,2t} \neq 0$ and hence $a_{2s-1.2s} = 1$. This proves that $C = B$ and hence $\{0, B\}^{\perp_k \perp_k} = \{0, B\}$. This completes the proof. □

Theorem 3.4. *Let k be a fixed positive integer such that $2 + 2k \leqslant \dim U$ and $k \geqslant 2$. Let T be a surjective mapping from $\Lambda^2 U$ to $\Lambda^2 W$ such that for any pair $A, B \in \Lambda^2 U$, we have $l(A - B) \leqslant k$ if and only if $l(T(A) - T(B)) \leqslant k$. Then there exist an invertible semilinear mapping $f : U \to W$, an element R in $\Lambda^2 W$ and a nonzero scalar c in F such that*

$$T(A) = c(\Lambda^2 f)(A) + R$$

for any A in $\Lambda^2 U$.

Proof. Since T preserves bivectors with bounded distance k in both directions, it follows that $\dim W \geqslant 2k + 2$. Using an argument similar to that of the first paragraph of the proof of Theorem 2.1, we can show that T is injective. Suppose that U is finite dimensional. Then from the third paragraph of the proof of Theorem 2.1, we see that $\dim U = \dim W$. Hence by Lemma 3.2, Lemma 3.3, Liu's theorem and Theorem 2.1, we obtain the result. □

When U is finite dimensional, Theorem 3.4 can be stated in matrix language as follows:

Corollary 3.5. *Let m, n and k be positive integers such that $m \geqslant 2k + 2$ and $k \geqslant 2$. Let T be a surjective mapping from $K_m(F)$ to $K_n(F)$ such that*

$$\rho(A - B) \leqslant 2k \text{ if and only if } \rho(T(A) - T(B)) \leqslant 2k.$$

Then $m = n$ and there exist an $n \times n$ invertible matrix P, an $n \times n$ alternate matrix R, an automorphism σ on F and a nonzero scalar λ such that

$$T(A) = \lambda P A_\sigma P^t + R$$

for any A in $K_n(F)$.

The following corollary is an immediate consequence of Corollary 3.5 and Liu's theorem [7].

Corollary 3.6. *Let n be a positive even integer $\geqslant 4$. Let $\tau : F \to F$ be a function such that only zero is mapped to zero. Then T is a surjective mapping from $K_n(F)$ onto itself such that*

$$\det(T(A) - T(B)) = \tau(\det(A - B))$$

for any A, B in $K_n(F)$ if and only if there exist an $n \times n$ invertible matrix P, an $n \times n$ alternate matrix R, an automorphism σ on F, and a nonzero scalar λ such that either

(i) $T(A) = \lambda P A_\sigma P^t + R$; or

(ii) $n = 4, T(A) = \lambda P \begin{pmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{pmatrix}_{\sigma} P^t + R, A = (a_{ij}) \in K_4(F),$

where in both cases $\tau = (\det(\lambda PP^t))\sigma$.

4. Automorphisms of graphs of symmetric matrices

Let $S_n(F)$ be the vector space of all $n \times n$ symmetric matrices over F. For any two positive integers n, s with $n \geqslant s$, let $\Gamma_s(S_n(F))$ denote the graph of all $n \times n$ symmetric matrices over F such that two distinct vertices A and B are joined by an edge if $\rho(A - B) \leqslant s$. Similarly, we use $\Gamma_k(K_n(F))$ to denote the graph of all $n \times n$ alternate matrices over F such that two distinct vertices A and B are joined by an edge if $\rho(A - B) \leqslant 2k$, where k is a fixed positive integer such that $2k \leqslant n$.

When (i) char $F \neq 2$ and $2 \leqslant k < n$, or (ii) $k = 1$ and $|F| > 2$, it is known [3,6,10,11] that every graph automorphism of $\Gamma_k(S_n(F))$ is of the form $A \mapsto \lambda PA_\sigma P^t + R$ where λ is a nonzero scalar in F, P is an $n \times n$ invertible matrix, σ is an automorphism on F, and R is a matrix in $S_n(F)$. However, the situation is different when char $F = 2$ and $2 \leqslant k$. In fact, when F is a perfect field of characteristic two, there are other types of graph automorphisms of $\Gamma_{2k}(S_n(F))$ when $n > 2k$.

Note that $K_n(F)$ is the second exterior power of F^n where $a \wedge b = a^tb - b^ta$, $a, b \in F^n$. Every element A in $K_n(F)$ of positive rank $2k$ is of the form $\sum_{i=1}^{k} u_{2i-1} \wedge u_{2i}$ for some linearly independent vectors $u_1, u_2, \ldots, u_{2k}$ in F^n and we use $[A]$ to denote the uniquely determined subspace $\langle u_1, u_2, \ldots, u_{2k}\rangle$.

The following lemma was proved in [12, Theorem 5.62], for the case $\rho(A) = 2$. Our proof for this case is different and shorter.

Lemma 4.1. *Let* char $F = 2$. *Let* $a, b \in F^n$ *and* $K \in K_n(F)$. *Let* $A = \begin{pmatrix} 0 & a+b \\ (a+b)^t & K \end{pmatrix}$ *and* $B = K + a^ta + b^tb$. *Then* $\rho(A) = \rho(B)$ *or* $\rho(A) = \rho(B) + 1$.

Proof. If $a = b$ or $K = 0$, the result is clear. Suppose now that $a \neq b$ and $K \neq 0$. Let $J = K + a \wedge b$. Then $B = J + (a + b)^t(a + b)$. Let $\rho(A) = 2s$. Then we have either $\rho(K) = 2s - 2$ or $\rho(K) = 2s$.

Case 1: $\rho(K) = 2s - 2$. We have $a + b \notin [K]$ and hence either (a) $\rho(J) = 2s - 2$, or (b) $\rho(J) = 2s$. Suppose that (a) holds. Let $P \in M_n(F)$ such that $(a + b)P = 0$ and $uP = u$ for every $u \in [K]$. Then $P^t(a \wedge b)P = 0$ and thus $P^tKP = P^tJP = K$. Hence $a + b \notin [J]$, otherwise $\rho(P^tJP) < 2s - 2$, a contradiction. This implies that $\rho(B) = 2s - 1$. Suppose that (b) holds. Then $[J] = [K] \oplus \langle a, b\rangle$. Hence $a + b \in [J]$ and this shows that $\rho(B) = 2s$ or $2s - 1$.

Case 2: $\rho(K) = 2s$. We have $a + b \in [K]$ and hence either (a)$\rho(J) = 2s - 2$, or (b) $\rho(J) = 2s$. Suppose that (a) holds. Then $[K] = [J] \oplus \langle a, b\rangle$ and hence $a + b \notin [J]$. This implies that $\rho(B) = 2s - 1$. Suppose that (b) holds. We shall show that $a + b \in [J]$. Suppose the contrary. Let $Q \in M_n(F)$ such that $(a + b)Q = 0$ and $uQ = u$ for every $u \in [J]$. Then $Q^tJQ = J = Q^tKQ$. Since $a + b \in [K]$, it follows that $\rho(Q^tKQ) < 2s$. However $\rho(J) = 2s$, a contradiction. Hence $a + b \in [J]$ and this implies that $\rho(B) = 2s$ or $2s - 1$. □

The following theorem shows that the graphs $\Gamma_k(K_{n+1}(F))$ and $\Gamma_{2k}(S_n(F))$ are isomorphic when F is a perfect field of characteristic two and n is an integer $\geqslant 2k + 1$. The case for $k = 1$ was proved in [12, Theorem 5.62].

Theorem 4.2. *Let F be a perfect field of characteristic two. Let n and k be two positive integers such that* $n \geqslant 2k + 1$. *Then the mapping* $\theta : K_{n+1}(F) \to S_n(F)$ *defined by*

$$\begin{pmatrix} 0 & a \\ a^t & K \end{pmatrix} \mapsto K + a^ta,$$

where a is an $1 \times n$ *matrix and K is an* $n \times n$ *alternate matrix, is a graph isomorphism between* $\Gamma_k(K_{n+1}(F))$ *and* $\Gamma_{2k}(S_n(F))$.

Proof. Since F is a perfect field of characteristic two, every matrix in $S_n(F)$ can be uniquely expressed as the sum of an alternate matrix and a symmetric matrix of rank $\leqslant 1$. This shows that the mapping θ is bijective. Let $A_1 = \begin{pmatrix} 0 & a \\ a^t & K_1 \end{pmatrix}$ and $A_2 = \begin{pmatrix} 0 & b \\ b^t & K_2 \end{pmatrix}$ be two matrices in $K_{n+1}(F)$ where K_1 and K_2 are $n \times n$ alternate matrices. Then

$$A_1 - A_2 = \begin{pmatrix} 0 & a+b \\ (a+b)^t & K_1 + K_2 \end{pmatrix}$$

and

$$\theta(A_1) - \theta(A_2) = (K_1 + K_2) + a^t a + b^t b.$$

By Lemma 4.1, we have $\rho(A_1 - A_2) \leqslant 2k$ if and only if $\rho(\theta(A_1) - \theta(A_2)) \leqslant 2k$. Hence θ is an isomorphism from $\Gamma_k(K_{n+1}(F))$ to $\Gamma_{2k}(S_n(F))$. □

Corollary 4.3. *Let F be a perfect field of characteristic two with at least four elements. Let n and k be two positive integers such that $n \geqslant 2k + 1 \geqslant 5$. Let $T : S_n(F) \to S_n(F)$ be a surjective mapping such that for any A, B in $S_n(F)$,*

$$\rho(A - B) \leqslant 2k \text{ if and only if } \rho(T(A) - T(B)) \leqslant 2k.$$

Then there exist an automorphism σ on F, a matrix R in $S_n(F)$ and an $(n+1) \times (n+1)$ invertible matrix $\begin{pmatrix} c & u \\ v & P \end{pmatrix}$ where P is an $n \times n$ matrix, such that

$$T(A) = Pa_\sigma^t v^t + va_\sigma P^t + PK_\sigma P^t + w^t w + R, \tag{3}$$

where $A = K + a^t a$ and $w = ua_\sigma^t v^t + (ca_\sigma + uK_\sigma)P^t$.

Proof. Let θ be the isomorphism from $\Gamma_k(K_{n+1}(F))$ to $\Gamma_{2k}(S_n(F))$ as mentioned in Theorem 4.2. Let $\varphi = \theta^{-1}$. We may assume that $T(0) = 0$. Then $L : K_{n+1}(F) \to K_{n+1}(F)$ defined by

$$L(\varphi(A)) = \varphi(T(A)), \quad A \in S_n(F),$$

is a surjective mapping such that for any C, D in $K_{n+1}(F)$,

$$\rho(C - D) \leqslant 2k \text{ if and only if } \rho(L(C) - L(D)) \leqslant 2k.$$

By Corollary 3.5, there exist an automorphism σ on F and an $(n+1) \times (n+1)$ invertible matrix $\begin{pmatrix} c & u \\ v & P \end{pmatrix}$ where P is an $n \times n$ matrix, such that

$$L\begin{pmatrix} 0 & a \\ a^t & K \end{pmatrix} = \begin{pmatrix} c & u \\ v & P \end{pmatrix} \begin{pmatrix} 0 & a \\ a^t & K \end{pmatrix}_\sigma \begin{pmatrix} c & v^t \\ u^t & P^t \end{pmatrix}$$
$$= \begin{pmatrix} 0 & w \\ w^t & Pa_\sigma^t v^t + va_\sigma P^t + PK_\sigma P^t \end{pmatrix},$$

where $w = ua_\sigma^t v^t + (ca_\sigma + uK_\sigma)P^t$. Hence

$$T(K + a^t a) = Pa_\sigma^t v^t + va_\sigma P^t + PK_\sigma P^t + w^t w.$$

This completes the proof. □

Remark 4.4. The mapping (3) in Corollary 4.3 preserves matrix pairs with bounded distance $2k$ in both directions.

Remark 4.5. Suppose that we impose as an additional condition to Corollary 4.3 that $T(A) - T(B)$ is alternate if $A - B$ is a rank 2 alternate matrix. Then we see that for any $n \times n$ alternate matrix K,

$$T(K) = PK_\sigma P^t + P(K_\sigma u^t u K_\sigma)P^t + R.$$

If $u \neq 0$, then it is easily seen that there exists a rank two alternate matrix J such that $J_\sigma u^t \neq 0$ and hence $T(J) - T(0)$ is not alternate, a contradiction. Hence $u = 0$ and $c \neq 0$. This implies that P is invertible and

$$T(K + a^t a) = Pa_\sigma^t v^t + va_\sigma P^t + PK_\sigma P^t + c^2 Pa_\sigma^t a_\sigma P^t + R.$$

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