New Renormalization Schemes for Conductivity Upscaling in Heterogeneous Media

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A fundamental problem in many fields of science and engineering is that of representing a heterogeneous conductivity field by a single 'effective' conductivity that characterizes the conditions on some suitable 'macroscopic' length scale. This process is known variously as 'upscaling' or 'homogenization' and common technological applications include the determination of effective hydraulic conductivities in petroleum andwater resources engineering and the determination of effective thermal conductivities, molecular diffusivities, and dielectric constants for natural and man-made composites. Common to these problems is the assumption that the transport is governed by a gradient type law on the 'microscale' (the smallest length scale considered) as well as on the 'macroscale' (the largest length scale considered). That is, the physical laws governing the transport are assumed to be scale invariant.

The assumption that the physics is invariant across the range of length scales considered is a most convenient one which allows for straightforward application of a large number of standard homogenization procedures. One such procedure is known as *renormalization*. Originally conceived in the field of quantum electrodynamics, the procedure was first applied to the problem of hydraulic conductivity upscaling by King (1989). The basic idea, which is illustrated in two spatial dimensions in Fig. 1, is the following. Given a square domain consisting of 2^n cells, each with a unique conductivity, a partial upscaling is performed by replacing each 2 × 2 block in the grid by a single cell with an appropriate representative block-conductivity. This gives a grid consisting of 2^{n-1} cells, and the procedure is then repeated recursively until only one macro-cell remains. The conductivity of this cell is the sought effective conductivity. Starting from this basic idea, a large number of different strategies have been developed (see e.g.,Green and Patterson 2007; Lunati et al. 2001; Renard and de Marsily 1997; Renard et al. 2000).

For the type of procedure described above, the most basic issue is that of determining the representative conductivity of a four-phase 2 × 2 block. A fundamental issue is here what boundary conditions should be applied to the block. Strictly speaking, the concept of 'effective' conductivity of random heterogeneous media applies only to volumes of an infinite extent. For finite-size volumes, for example 2 × 2 blocks, the 'effective' conductivity will depend on the boundary conditions. In classic renormalization procedures, the boundary conditions are usually of the mixed type. In the two-dimensional case, no-flow Neumann boundary conditions are enforced on two opposite edges while the Dirichlet boundary conditions are enforced on the other two edges (see Fig. 2). Similarly, in three dimensions, Dirichlet conditions are applied on two opposite faces while Neumann boundary conditions are enforced on all other faces. In ensemble averaging procedures, pure Dirichlet and Neumann boundary conditions are known to result in upper and lower bounds, respectively, on the effective conductivity while mixed boundary conditions of the type shown in Fig. 2 result in an 'effective' conductivity that falls in between these two bounds Karim and Krabbenhoft 2010; Ostoja-Starzewski and Schulte 1996). Considering the approximate nature of renormalization, the use of mixed boundary conditions thus appears to be quite reasonable. In the following, all results on block-conductivities assume mixed boundary conditions of the kind shown in Fig. 2.

In the context of hydraulic conductivity upscaling, renormalization was first considered by King (1989) who devised a finite-difference type solution to the calculation of blockconductivities. In the two-dimensional case this leads to a closed-form expression for the block-conductivity. For the three-dimensional case similar arguments can be used although the equivalent closed-form expression is somewhat more problematic to derive and rather expensive to compute (Green and Patterson 2007). The immediate question that arises concerns the accuracy of the block-conductivity solution. Surprisingly, although the relevant two-dimensional solution has been available for some years (Craster and Obnosov 2001; Milton 2001; Mortola and Steffe 1985), it has not to our knowledge been employed in renormalization schemes. The first aim of the present paper is therefore to use this solution in a conventional 2 × 2 renormalization scheme as described above. Second, inspired by the structure of the exact two-dimensional block-conductivity solution, we propose an approximate expression for the block-conductivity of a three-dimensional 2×2×2 block. The new schemes are validated against both analytical and numerical (finite element/difference type) solutions.

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