A RÈLATIVISTIC WINDOWED FOURIER TRANSFORM

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Abstract

Drawing upon some earlier work on coherent states of the Poincare' group in 1-space and 1-time dimensions, we use these states to define a relativistic windowed Fourier transform, as relativistic extension of the usual windowed Fourier transform. We discretize the resulting transform and obtain conditions under which the discretized transform can be used to reconstruct arbitrary square integrable functions. We present some numerical and graphical exercises to illustrate the theory, as well as to compare the relativistic windowed Fourier transform with the orthodox windowed Fourier transform.

1. Introduction

It is well known that the windowed Fourier transform (WFT), also called short-time Fourier transform or the Gabor transform [1], owes its origin to the *coherent states* (CS) of the Weyl-Heisenberg group. For a function $f \in L^2(R)$, its windowed Fourier transform is

$$(W_\eta f)(q,p) = \langle \eta_{q,p} \mid f \rangle = \int_R e^{-ikq} \eta(k-p) f(k) dk, \qquad (1)$$

where $\langle | \rangle$, denotes the scalar product in $L^2(R)$ and $\eta_{q,p}$, $(q,p) \in R^2$, are the coherent states of Weyl-Heisenberg group, built out of η :

$$\eta_{q,p}(k) = e^{-ikq} \eta(k-p). \qquad (2)$$

Thus the WFT maps a function $f \in L^2(R)$, to a function of two variables q and p, (which could be interpreted as time and frequency). Moreover, as a linear map $W_\eta : L^2(R) \rightarrow L^2(R^2)$, is a Hilbert space isometry. Since the functions $\eta_{q,p}$ also appear as the CS of the Galilei group [2] an analogous transform, using the CS of the Poincare' group in 1-space and 1-time dimensions could naturally be termed a *relativistic windowed Fourier transform* (RWFT). Large classes of such CS have been obtained earlier in [3] and for the Poincare' group in 3-space and 1-time dimensions in [4]. A standard problem in signal analysis is to take a continuous transform such as the function $(q,p) \mapsto (W_\eta f)(q,p)$ and to discretize it, in other words, to construct algorithms for reconstructing $W_\eta f$ from a knowledge of its values at a discrete set of points $(q_i, p_i), i = 1,2,3,..$. In this paper we address ourselves to this particular problem for the relativistic windowed Fourier transform. As just noted, this transform is built out of the CS of the Poincare' group $P_+^\uparrow(1,1)$ in 1-space and 1-time dimensions. A fairly general technique exists [2,5] for constructing families of coherent states of semi-direct product type groups, of which $P_+^\uparrow(1,1)$ is an example. Given a group of the type $G = R^n \otimes S$, where S is a closed subgroup of $GL(n,R)$, one looks at orbits of fixed vectors $\vec{k}_0$ in the dual of R^n. Such an orbit O is an m-dimensional manifold $(m \leq n)$.

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Let S_0 be the stability subgroup of $\vec{k}_0$, so that $O \cong S / S_0$ and let $T\vec{k}_0 O$ be the tangent space of O at $\vec{k}_0$. The annihilator of $T\vec{k}_0 O$ in R^n is a subgroup N_0 of the abelian group R^n. The quotient space $\Gamma = G / H$, where $H = N_0 \otimes S_0$ can be identified with an orbit of G under the coadjoint action. Suppose now that $g \mapsto U(g)$ is a unitary irreducible representation (UIR) of G, on a Hilbert space H, which is induced from the representation $\Re L$, of $R^n \otimes S_0$, where for $(\vec{x}, s) \in R^n \otimes S_0$,

$$\Re L(\vec{x}, s) = \Re(\vec{x}) L(s) = e^{-i\vec{k}_0 . \vec{x}} L(s). \qquad (3)$$

Here L is a UIR of S_0 on some Hilbert space χ. In order to construct CS one next looks for sections $\sigma : \Gamma \to G$ and vectors $\eta^i \in \mathrm{H}, i = 1,2,3....N$, such that

$$\sum_{i=1}^{N} \int_{\Gamma} | \eta^i_{\sigma(\gamma)} \rangle \langle \eta^i_{\sigma(\gamma)} | \, d\nu(\gamma) = A_{\sigma,} \qquad (4)$$

where

$$\eta^i_{\sigma(\gamma)} = U(\sigma(\gamma)) \eta^i, \ \gamma \in \Gamma, i = 1,2,3....N,$$

$d\nu$ is the invariant measure on Γ and A_σ is a bounded linear operator on H with bounded inverse. If such a section σ and set of a vectors η^i exists, the vectors $\eta^i_{\sigma(\gamma)}$,

$\gamma \in \Gamma, i = 1,2,3...N$, are then called coherent states for the representation *U*. Equation (4) now guarantees that any vector $\phi \in \mathrm{H}$ can be written as a linear combination of the vectors $\eta^i_{\sigma(\gamma)}$. When $G = P_+^{\uparrow}(1,1) = R^2 \otimes L_+^{\uparrow}(1,1)$, where $L_+^{\uparrow}(1,1)$ is the proper Lorentz group in 1-space and 1-time dimensions, one obtains in this manner relativistic CS. It turns out that now N = 1, and the function

$$\gamma \mapsto \Phi(\gamma) = \langle \eta_{\sigma(\gamma)} | \phi \rangle, \ \ \gamma \in \Gamma, \qquad (5)$$

is called the relativistic Fourier transform of the vector $\phi \in \mathrm{H}$. We shall make the form of Φ more explicit later. Equation (4) is also known as the frame condition for the CS, $\eta^i_{\sigma(\gamma)}$. In the special case where $A_\sigma = \lambda I, \lambda \in R$, I is the identity operator on H, the frame is said to be *tight*.

The CS, $\eta^i_{\sigma(\gamma)}$, form an *overcomplete* set of vectors in the Hilbert space H, and it then becomes interesting to find a discrete set of points, $\gamma_j, j = 1,2,3...$, such that the vectors $\eta^i_{\sigma(\gamma_j)}$ still span the space H, and which satisfy a *discrete frame condition*

$$\sum_{i,j} | \eta^i_{\sigma(\gamma_j)} \rangle \langle \eta^i_{\sigma(\gamma_j)} | = T, \qquad (6)$$

where again both T and T^{-1} are bounded operators on H. Once this is achieved, any vector $\phi \in \mathrm{H}$ can be written in terms of the frame vectors $\eta^i_{\sigma(\gamma_j)}$ in the manner

$$\phi = \sum \Phi^i(\gamma_j) T^{-1} \eta^i_{\sigma(\gamma_j)}, \qquad (7)$$

$$\text{where} \ \ \Phi^i(\gamma_j) = \langle \eta^i_{\sigma(\gamma_j)} | \phi \rangle. \qquad (8)$$

Equation (7) is called a *reconstruction formula.* We derive such reconstruction formula for the RWFT's and as a practical application we reconstruct the function e^{-t^2} using different window functions. Of course, in each case the sum in (7) is truncated after a finite number of terms and we graphically compare these approximations with the original function. It is observed that for a smooth window the reconstruction scheme does a better job than that for a non-smooth window. However different sections play more or less the same role under similar situations. Finally, we compare our reconstruction scheme with the one obtained using the usual window Fourier transform (2), by reconstructing a function using two methods.

The two sets of reconstructed values of the function turn out to in close conformity.

2. Notation and Formalism

Elements of the Poincare' group $P_+^\uparrow(1,1)$ are denoted by $g=(a;\Lambda)$, where $a=(a_0,\vec{a})\in R^2$ is a space-time translation and Λ a Lorentz boost. Let V_m^+ denote the forward mass hyperbola,

$$V_m^+ = \{p=(p_0,\vec{p})\in R^2 \mid p_0^2-\vec{p}^2=m^2, p_0>0\} \qquad (9)$$

The matrix Λ is parametrized by a vector

$p=(p_0,\vec{p})\in V_m^+$ as

$$\Lambda_p = \frac{1}{m}\begin{pmatrix} p_0 & \vec{p} \\ \vec{p} & p_0 \end{pmatrix}. \qquad (10)$$

We work with the unitary irreducible representation U of $P_+^\uparrow(1,1)$, on the Hilbert space $\mathrm{H}=L^2(V_m^+, d\vec{k}/k_0)$, (11)
for a given $g=(a;\Lambda_p)\in P_+^\uparrow(1,1)$, by

$$(U(g)\eta)(k)=e^{ik\cdot a}\eta(\Lambda_p^{-1}k), \forall \eta\in \mathrm{H}, \qquad (12)$$

where $k\cdot a=k_0a_0-\vec{k}\cdot\vec{a}$.

Let $\Gamma=P_+^\uparrow(1,1)/T$, where T is the subgroup of time translations. Then Γ has global coordinatization $(\vec{q},\vec{p})\in R^2$ in terms of which the left invariant measure is $d\vec{q}d\vec{p}$ [3]. The action of $P_+^\uparrow(1,1)$ on Γ in these coordinates is given by

$$(a,\Lambda_p)\cdot(\vec{q},\vec{p})=(\vec{q}',\vec{p}'), \qquad (13)$$

$$\text{with} \quad \left.\begin{array}{l} \vec{p}'=\Lambda_k \vec{p} \\ \vec{q}'=\frac{1}{p_0}\left[p_0\vec{q}+m(\vec{\Lambda}_p^{-1}a)\right] \end{array}\right\} \qquad (14)$$

We work with a class of sections, called *affine sections*, obtained by starting with the section

$$\begin{aligned} &\sigma_o:\Gamma\to P_+^\uparrow(1,1),\\ &\sigma_0(\vec{q},\vec{p})=((0,\vec{q}),\Lambda_p),\\ &p=(\sqrt{|\vec{p}|^2+m^2},\vec{p}), \end{aligned} \qquad (15)$$

and then defining a general section as

$$\sigma(\vec{q},\vec{p})=\sigma_0(\vec{q},\vec{p})((f(\vec{q},\vec{p}),0),I), \qquad (16)$$

where f is a smooth real-valued function of the type

$$f(\vec{q},\vec{p})=\varphi(\vec{p})+\vec{q}\vartheta(\vec{p}), \qquad (17)$$

where both φ and ϑ are real-valued functions of $\vec{p}$ alone. With this,

$$\sigma(\vec{q},\vec{p})=(\hat{q},\Lambda_p), \qquad (18)$$

$$\hat{q}_0=p_0f(\vec{q},\vec{p})/m, \qquad (19)$$

$$\hat{\vec{q}}=\vec{q}+\vec{p}f(\vec{q},\vec{p})/m \qquad (20)$$

Since φ plays an inessential role for our present purposes, we set $\varphi=0$. Let us rewrite (19) as

$$\hat{q}_0=\beta(\vec{p})\hat{\vec{q}} \qquad (21)$$

with $\beta(\vec{p})=p_0\vartheta(\vec{p})/(m+\vec{p}\vartheta(\vec{p}))$. (22)

Solving for
$\vartheta(\vec{p})=m\beta(\vec{p})/(p_0-\vec{p}\beta(\vec{p}))$ (23)

Let us also introduce the *dual* vector fields β^*,ϑ^*,

$$\beta^*(\vec{q},\vec{p})=(\vec{p}-p_0\beta(\vec{p}))/(p_0-\vec{p}\beta(\vec{p})) \qquad (24)$$

$$\vartheta^*(\vec{p})=m\beta^*(\vec{p})/(p_0-\vec{p}\beta(\vec{p})) \qquad (25)$$

Note that $\beta^{**}=\beta$, $\vartheta^{**}=\vartheta$, and
$\vartheta(\vec{p})=(1/m)[\vec{p}-p_0\beta^*(\vec{p})$. (26)

The coherent states of $P_+^\uparrow(1,1)$, for an arbitrary section $\sigma(\vec{q},\vec{p})$, are now the set of vectors

$$\eta_{\sigma(\vec{q},\vec{p})}(k) = e^{-iX_p(k)\vec{q}}\eta(\Lambda_p^{-1}k) \tag{27}$$

where $X_{\vec{p}}(\vec{k}) = \vec{k} - (\Lambda_p^{-1}k)_0\vartheta(\vec{p})$ (28)

In terms of these CS, the relativistic Fourier transform $\Phi(\vec{q},\vec{p})$ of a function $\phi \in L^2(V_m^+, d\vec{k}/k_0)$ becomes (see (5)):

$$\Phi(\vec{q},\vec{p}) = \int_{V_m^+} \exp[i\{\vec{k}\cdot\vec{q} - (\Lambda_p^{-1}k)_0\vartheta(\vec{p})\vec{q}\}] \times\overline{\eta}(\Lambda_p^{-1}k)\phi(k)d\vec{k}/k_0 \tag{29}$$

where "-" indicates complex conjugation.

3. Periodization of Compactly Supported Functions

Let $Y: R \to C$ be a compactly supported function, with support [a, b] and the length of the support $\ell = b - a < \infty$. Let $\widetilde{Y}: R \to C$ be a periodic function with period ℓ such that

$$\widetilde{Y}(x) = Y(x), \forall x \in [a,b], \tag{30}$$

$$\widetilde{Y}(x+\ell) = \widetilde{Y}(x), \forall x \in R \tag{31}$$

Then $\widetilde{Y}$ has the Fourier series decomposition

$$\widetilde{Y}(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx/\ell}, \forall x \in R, \tag{32}$$

and

$$Y(x) = \sum c_n e^{i2\pi nx/\ell}, \forall x \in [a,b], \tag{33}$$

where, $c_n = (1/\ell)\int_a^b \widetilde{Y}(x)e^{-i2\pi nx/\ell}dx$ (34)

For $j \in Z$ (the set of all integers), let $p_j = (p_{j0}, \vec{p}_j)$ be a discretization of p. Then for an arbitrary section $\sigma(\vec{q},\vec{p})$ and $\ell, j \in Z$, we write the discretized version of the coherent states in (27) as

$$\eta_{\ell,j}(k) = e^{-iX_j(k)\vec{q}_{\ell,j}}\eta(\Lambda_j^{-1}k), \vec{q}_{\ell,j} = \Delta\vec{q}_j\ell, \tag{35}$$

where $\Delta\vec{q}_j > 0$ is to be fixed later, $X_j(\vec{k})$ and Λ_j are respectively the discretized forms of $X_p(\vec{k})$ and Λp. Let $a = (a_0, \vec{a}) \in V_m^+$ and $b = (b_0, \vec{b}) \in V_m^+$ and suppose that $\eta(k) = 0$ if $\vec{k} \notin [\vec{a}, \vec{b}]$, i. e., the length of the support of $\eta(k)$ is $\vec{b} - \vec{a}$. Then the length of the support of $\eta(\Lambda_j^{-1}k)$ is

$$[(b_0 - a_0)p_j + (\vec{b} - \vec{a})p_{j0}]/m \tag{36}$$

Let $\eta(\Lambda_j^{-1}k) = \widetilde{\eta}(X_j(\vec{k}))$. Then the length of the support L_j of $\widetilde{\eta}(X_j(\vec{k}))$ is given by

$$L_j = [(b_0 - a_0)p_j + (\vec{b} - \vec{a})p_{j0}]/m - -(b_0 - a_0)\vartheta(p_j) \tag{37}$$

After discretization we can write the frame operator (6) and the relativistic Fourier transform in (29) in the following forms:

$$T = \sum_{\ell,j=\infty}^{\infty} |\eta_{\ell,j}\rangle\langle\eta_{\ell,j}| \tag{38}$$

$$\langle\eta_{\ell,j}|\phi\rangle = \int e^{iX_j(\vec{k})\vec{q}_{\ell,j}}\eta(\overline{\Lambda}_j^{-1}k)\phi(k)d\vec{k}/k_0 \tag{39}$$

4. The Frame Operator and the Reconstruction Formula

To examine the convergence of the operator sum in (38) we consider, for arbitrary $\phi, \psi \in \mathrm{H}$, the formal sum

$$\langle\phi|T\psi\rangle = \sum\langle\phi|\eta_{\ell,j}\rangle\langle\eta_{\ell,j}\psi\rangle$$

$$= \sum_{\ell, j=\infty}^{\infty} \iint_{V_m^+} e^{[i\{X_j(\vec{k})-X_j(\vec{k}')\}q_{\ell,j}]} \overline{\phi}(k)\eta(\Lambda_j^{-1}k)$$
$$\times \overline{\eta}(\Lambda_j^{-1}k')\psi(k')(d\vec{k}/k_0)(d\vec{k}'/k_0') \tag{40}$$

Letting $\eta(\Lambda_j^{-1}k) = \tilde{\eta}(X_j(\vec{k}))$, $\Delta\vec{q}_j = 2\pi / L_j$, changing the variables $\vec{k}' \to X_j(\vec{k})$ and using the following relation of the Dirac delta function

$$1/L \sum_{n=-\infty}^{\infty} e^{i2\pi n 9X-X')/l} = \delta(x-x')$$

we obtain

$$\langle \phi | T\psi \rangle = \int_{V_m^+} (d\vec{k}/k_0)\phi(k)\psi(k)$$
$$\times \sum_{j=-\infty}^{\infty} (1/L_j)(k_0 - (\vec{\Lambda}_j^{-1}k)\vartheta(p_j)) | \eta(\Lambda_j^{-1}k)|^2 . \tag{41}$$

Hence, defining T as the multiplication operator:

$(T\phi)(k) = T(k)\phi(k)$, with

$$T(k) = (1/L_j)(k_0 - (\vec{\Lambda}_j^{-1}k)\vartheta(p_j)) | \eta(\Lambda_j^{-1}k)|^2, \tag{42}$$

we see that the boundedness and invertibility and strict positivity of the function *T(k)*. For $t, \theta_j, \theta_a, \theta_b \in R$, let

$$\begin{aligned} &k_0 = m\cosh(t), \vec{k} = m\sinh(t), \\ &p_0 = m\cosh(\theta_j), p_j = m\sinh(\theta_j), \\ &a_0 = m\cosh(\theta_a), \vec{a} = m\sinh(\theta_a), etc. \end{aligned} \tag{43}$$

we have,

$$T(t) = 2\sinh\big((\theta_b - \theta_a)/2\big)$$
$$\sum_{j=-\infty}^{\infty} \{[\cosh(((\theta_b + \theta_a)/2) + \xi(j\theta_0))]/$$
$$[\cosh(t - j\theta_0 + \xi(j\theta_0))]\} | \eta(t - j\theta_0)|^2, \tag{44}$$

where we have written, $\theta_j = j\theta_0, \theta_0 > 0,$ (fixed) we call it *step size*, ξ is an arbitrary real-valued function, which represents, for different values of its arguments, different sections. For example, $\xi(x) = x$, it represents *Galilean section*, $\xi(x) = 0$,
the *Lorentz section* and $\xi(x) = (1/2)x$, the *symmetric section*. For any compactly supported function η, with length of support L, the sum in (44) contains at best $L/\theta_0 + 1$ terms, i.e., it is a finite sum. We observe that each term in the sum is positive and bounded for any $t \in R$ and consequently, the function $T(t)$ is strictly positive and bounded, hence T is bounded with bounded inverse. Thus the discretized version of $\eta_{\ell,j}$ of the coherent states in (27) form a *frame*, more appropriately, a *discrete frame*. For the *Lorentz* and *symmetric* section and some other condition on the window function η, this frame could be *tight*.

After discretization, the coherent states in (27) takes the form

$$\eta_{\ell,j}(t) = \eta(t - j\theta_0)$$
$$\times \exp[-i\pi\ell\{\sinh(t - j\theta_0 + \xi(j\theta_0))\}/$$
$$\{\sinh((\theta_b - \theta_a)/2)\cosh((\theta_b + \theta_a)/2 +$$
$$\xi(j\theta_0))\}] \tag{45}$$

Using (45) we can write scalar product in (39) as

$$\langle \eta_{\ell,j} | \phi \rangle = \int_{-\infty}^{\infty} \exp[i\pi\ell\{\sinh(t - j\theta_0 + \xi(j\theta_0))\}$$
$$/\{\sinh((\theta_b - \theta_a)/2)\cosh((\theta_b + \theta_a)/2 +$$
$$\xi(j\theta_0))\}]\overline{\eta}(t - j\theta_0)\phi(t)dt \tag{46}$$

which we call the *discretized relativistic windowed Fourier transform* of the function $\phi(t)$ for the window-function $\eta(t)$. Using (38) we can write

$$\phi(t) = \sum_{\ell, j=-\infty}^{\infty} \langle \eta_{\ell,j} | \phi \rangle [T(t)]^{-1} \eta_{\ell,j}(t) \tag{47}$$

for any $\phi \in R$. We (47) the *reconstruction formula*. In the next section, we use the frame operator and the reconstruction formula to reconstruct some function in H numerically.

5. Reconstruction of Functions

We here present some specific reconstruction results. As a prototype, we take the function $\phi(t) = e^{-t^2}$ and first reconstruct it for a triangular window function

$$\eta(t) = \begin{cases} 1+t, if -1 \le t < 0 \\ 1-t, if 0 \le t < 1. \end{cases} \quad (48)$$

Since [thesis] that the different sections play the more or less same role in the reconstruction scheme, here we just use the Galilean section.

For this reconstruction, we have retained the terms corresponding to $-50 \le \ell \le 50$ in the reconstruction formula and took the step size $6_0 = 0.025$. The reconstructed function along with the original one is shown in the Figure 1 from where we see that the accuracy of the reconstruction is not very high specially around the origin for the non smooth window function.

Then we reconstructed the same function $\phi(t) = e^{-t^2}$ for a smooth window function

$$\eta(t) = \begin{cases} (1-t^2)^{10}, if - \le t \le 1 \\ 0, otherwise. \end{cases} \quad (49)$$

Here we have taken the same step size as in the case of non-smooth window function and retained the smaller range of terms, $-20 \le \ell \le 20$. The graph of the reconstructed function along with the original one is shown in the Figure 2. From the Figure 2, we observe that the reconstruction scheme goes very well with a smooth window function. We also reconstructed a discontinuous function

$$\phi(t) = \begin{cases} t, if -2 \le t \le 2 \\ 0, otherwise \end{cases} \quad (50)$$

The graph of the reconstructed function and its original counter part are shown in the Figure 3. From this graph we see that in the vicinity of a point t_0 where $\phi(t)$ is discontinuous, the reconstructed function starts oscillating and the accuracy of approximation deteriorates. In Fourier analysis, this fact is known as the *Gibbs' phenomenon* [7]. We also observe that at any such point t_0, $\phi(t_0)$ converges to

$$\left(\phi(t_0^+) + \phi(t_0^-)\right)/2,$$

where $\phi(t_0^+)$ and $\phi(t_0^-)$ are respectively the right- and left-hand limit of $\phi(t)$ at $t = t_0$.

6. Comparison with the Windowed Fourier Transform

As a final example, we discretize the coherent states of the Weyl-Heisenberg group and came up with the corresponding frame operator and reconstruction formula. The discretized version of the coherent states of the Weyl-Heisenberg is given by:

$$\eta_{m,n}(x) = e^{imp_0x}\eta(x - nq_0) \quad (51)$$

where $p_0, q_0 > 0$ and m, n are integers and p_0, q_0 must satisfy the condition

$$p_0 \cdot q_0 \le 2\pi \quad (52)$$

The condition (53) is necessary for $\eta_{m,n}(x)$ to be complete and to form a frame [7,8,9]. For the present case, the frame operator and the reconstruction formula are give by:

$$T(x) = (2\pi / p_0)\sum_{n=-\infty}^{\infty} |\eta(x - nq_0)|^2 \quad (53)$$

and

$$\psi(x) = [1/T(x)] \sum_{m,n=-\infty}^{\infty} \langle \eta_{m,n} | \psi \rangle \eta_{m,n}(x) \quad (54)$$

Now we would like to reconstruct the function $f(t) = \cosh(t)e^{-\sinh^2(t)}$ using RWFT and WFT. In this case, we took the step size $6_0 = 0.01, p_0 = \pi, q_0 = 0.01$ and $-20 \le m, \ell \le 20$. The graphs of the reconstructed functions along with the original one are shown in the Figure 4 from where we see that the reconstruction scheme yields almost the same results, for this function, both for the WFT and RWFT.

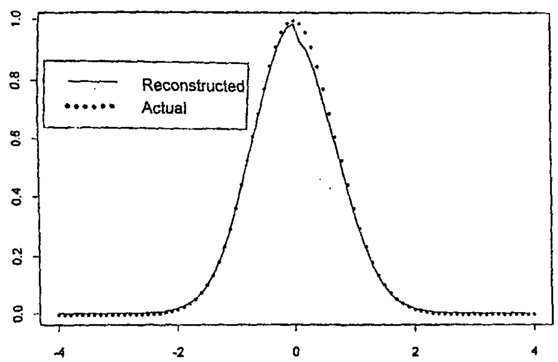

Figure 1. Reconstruction of e^{-t^2} for the Galilean section and the triangular window.

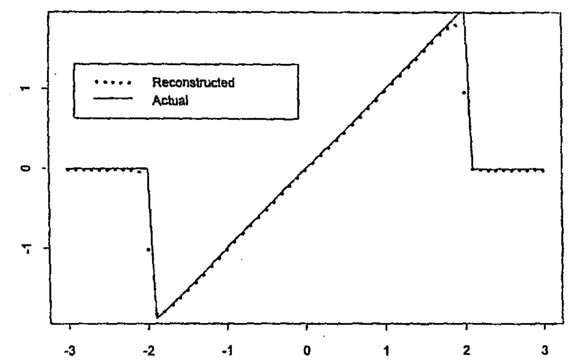

Figure 3. Reconstruction of a discontinuous function for the Galilean section and the smooth window.

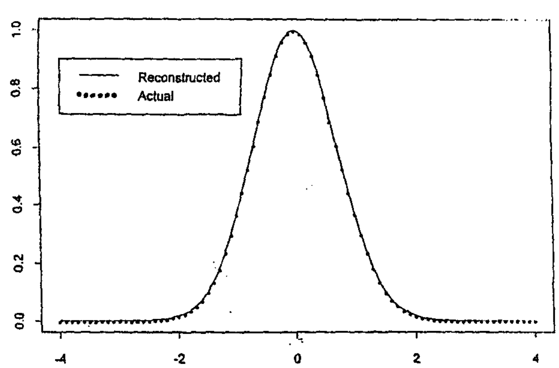

Figure 2. Reconstruction of e^{-t^2} for the Galilean section and the smooth window.

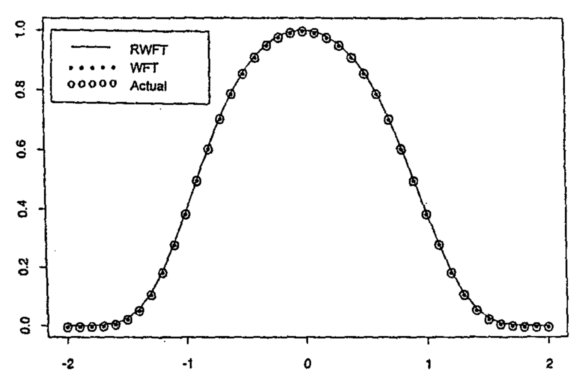

Figure 4. Reconstruction of $\cosh(t)e^{-\sinh(t)^2}$ using RWFT and WFT.

7. Discussion

The illustrative computations presented in this paper do not necessarily establish the superiority, or otherwise, of the relativistic windowed Fourier transform over the conventional windowed Fourier transform. However, the RWFT demonstrates how transforms may be obtained using a large class of semi-direct product type groups and indeed, it points up the richness and variety of the transforms so available. Work presently underway attempts to adapt the transformation which is employed to analyze a particular signal, to geometry of the problem, as reflected in the symmetry group that is used for constructing the CS. Furthermore, such an analysis is expected to provide information on the classes of functions which could be analyzed most effectively using particular types of transforms.

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