On Approximation and Biasness of Error Concentration Parameter for Circular Functional Model

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(Papers presented at the ***International Conference on Mathematics: Trends and Developments*** held on 27-30 December 2007 in Cairo, Egypt)

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Abstract: This paper introduce a simple procedure for obtaining estimate of the concentration parameter of the von Mises distribution for the circular functional relationship model. The estimation of parameters for this model assume the ratio of the error concentration parameter is known and the available method can only be applied when the ratio is equal to one or when we have an equal error concentration parameter. This new proposed procedure is based on the asymptotic power series of the modified Basel function of the first kind and order zero and it is shown that it can be applied for any value of the ratio of the error concentration parameter.

1. Introduction

Suppose the variables X and Y are related by $Y = \alpha + \beta X$, where both the X and Y are observed with errors. This model comes under the errors-in-variables model (EIVM). The EIVM differs from the ordinary or classical linear regression model in that the true independent variables or the explanatory variables are not observed directly, but are masked by measurements error. If X is a mathematical variables, this termed as a linear functional relationship model, and if X is a random variables, then this is termed a linear structural relationship model between X and Y. This paper consider the relationship when both variables are circular which takes values on the circumference of a circle, i.e. they are angles in the range $(0, 2\pi)$ radians or $(0^\circ, 360^\circ)$. Some of the examples are the wind and wave direction data measured by two different methods, the anchored wave buoy and HF radar system. Since the variables are circular we refer the model as the linear circular functional relationship model or circular functional model and as an analogy to the linear functional relationship model, we assume the errors of circular variables X and Y are independently distributed and follow the von Mises distribution with mean zero and concentration parameters κ and ν, respectively. For any circular random variable θ, it is said to have a von Mises distribution if its p.d.f is given by $g(\theta;\mu_0,\kappa)=\frac{1}{2\pi I_0(\kappa)}\exp\{\kappa\cos(\theta-\mu_0)\}$, where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. The parameter μ_0 is the mean direction while the parameter κ is described as the concentration parameter. This research present the mathematical approach on how to find the estimation of concentration parameters for the unreplicated linear circular functional relationship model assuming the ratio of error concentration parameters, λ is known. We have also shown that by using the approximation and asymptotic properties of Bassel function we can find the estimate of error concentration parameter for any value of λ.

2. The model

Suppose x_i and y_i are observed values of the circular variables X and Y, respectively, thus $0 \le x_i, y_i < 2\pi$, for $i = 1, \ldots, n$. For any fixed X_i, we assume that the observations x_i and y_i (which are unreplicated) have been measured with errors δ_i and ε_i, respectively and thus the full model can be written as

$$x_i = X_i + \delta_i \text{ and } y_i = Y_i + \varepsilon_i, \text{ where}$$
$$Y_i = \alpha + \beta X_i \pmod{2\pi}, \text{ for } i = 1, 2, \ldots, n.$$

We also assume δ_i and ε_i are independently distributed with (potentially different) von Mises distributions, that is $\delta_i \sim VM(0,\kappa)$ and $\varepsilon_i \sim VM(0,\nu)$. Suppose we assume that the ratio of the error concentration parameters, that is $\frac{\nu}{\kappa}=\lambda$ is known. Then the log likelihood function is given by

$$\log L(\alpha,\beta,\kappa,X_1,\ldots,X_n;\lambda,x_1,\ldots,x_n,y_1,\ldots,y_n)=$$
$$-2n\log(2\pi)-n\log I_0(\kappa)-n\log I_0(\lambda\kappa)+\kappa\sum\cos(x_i-X_i)+\lambda\kappa\sum\cos(y_i-\alpha-\beta X_i).$$

In the next section we will show how to estimate the error concentration parameter κ when $\lambda = 1$. Further we will extend the case by using the asymptotic properties of the Bessel function to estimate κ for any value of λ .

3. Estimation of the concentration parameter κ

By setting $\frac{\partial \log L}{\partial \kappa} = 0$ of the log likelihood function we get the equation

$$A(\kappa) + \lambda A(\lambda\kappa) = \frac{1}{n}\left\{\sum \cos(x_i - \hat{X}_i) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i)\right\}. \qquad (1)$$

The approximation given by Dobson [2], that is

$$A^{-1}(w) = \frac{9 - 8w + 3w^2}{8(1-w)},$$

or by Best & Fisher [3], that is

$$A^{-1}(w) = \begin{cases} 2w + w^3 + \dfrac{5w^5}{6}, & w < 0.53 \\ -0.4 + 1.39w + \dfrac{0.43}{1-w}, & 0.53 \le w < 0.85 \\ \dfrac{1}{w^3 - 4w^2 + 3w}, & w \ge 0.85 \end{cases}$$

to estimate κ in (1) can only be used for the case when $\lambda = 1.0$. In this section we show that by using the asymptotic properties of the Bessel function we can find an estimate of κ for any value of λ.

From the asymptotic power series for the Bessel functions $I_0(r)$ and $I_1(r)$ in [1], we have,

$$A(r) = \frac{I_1(r)}{I_0(r)} = 1 - \frac{1}{2r} - \frac{1}{8r^2} - \frac{1}{8r^3} + 0(r^{-4}) \qquad (2)$$

Simplifying the equation (1) using (2) we have the expression approximately given by

$$8(1+\lambda - c)\kappa^3 - 8\kappa^2 - (1+\frac{1}{\lambda})\kappa - (1+\frac{1}{\lambda^2}) = 0, \qquad (3)$$

where

$$c = \frac{1}{n}\left\{\sum \cos(x_i - \hat{X}_i) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i)\right\}.$$

It can be shown that the above cubic equation in κ, i.e. equation (3) has only one positive real root and two complex roots, giving $\hat{\kappa}$ as the positive real root by using the following procedure.

Firstly, we find a suitable substitution such that the equation can be transformed into the form $x^3 + px + q = 0$. This can be done by define $D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ and we follow the following rules [5],

a) if $D > 0$, there exist one real and two complex roots,

b) if $D = 0$, there exist three real roots (at least two are equal), and

c) if $D < 0$, there exist three distinct real roots.

The following step is that, for $D > 0$, let r_1, r_2 and r_3 be the roots, then

$$r_1 = u + v,$$

$$r_2 = -\left(\frac{u+v}{2}\right) + \left(\frac{u-v}{2}\right)\sqrt{3}i, \text{ and}$$

$$r_3 = -\left(\frac{u+v}{2}\right) - \left(\frac{u-v}{2}\right)\sqrt{3}i.$$

where

$$u = \left(\frac{-q}{2} + \sqrt{D}\right)^{1/3} \text{ and } v = \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3}.$$

Our aim is to solve the following equation for κ

$$8(1+\lambda-c)\kappa^3 - 8\kappa^2 - (1+\frac{1}{\lambda})\kappa - (1+\frac{1}{\lambda^2}) = 0$$

or

$$a_0\kappa^3 + a_1\kappa^2 + a_2\kappa + a_3 = 0,$$

where

$$a_0 = 8(1+\lambda-c) > 0, \text{ since } \lambda > 0$$

$$\text{and } c = \frac{1}{n}\left(\sum \cos(x_i - \hat{X}_i) + \lambda \sum \cos(y_i - \hat{\alpha} - \hat{\beta}\hat{X}_i)\right) < (1+\lambda)$$

$$a_1 = -8, \quad a_2 = -\left(1+\frac{1}{\lambda}\right), \text{ and } a_3 = -\left(1+\frac{1}{\lambda^2}\right).$$

A suitable substitution is $k = y - \left(\frac{a_1}{3a_0}\right)$, which gives $y^3 + py + q = 0$, where

$$p = \frac{3a_0a_2 - a_1^2}{3a_0^2}, \text{ and}$$

$$q = \frac{2a_1^3 - 9a_1a_2a_0 + 27a_3a_0^2}{27a_0^3}.$$

These coefficients may be written as

$$p = -\left(\frac{3\Delta(1+\lambda)+8\lambda}{24\lambda\Delta^2}\right), \text{ and}$$

$$q = -\left(\frac{16\lambda^2 + 9\lambda(\lambda+1)\Delta + 27(\lambda^2+1)\Delta^2}{216\lambda^2\Delta^3}\right),$$

where

$$\Delta = 1 + \lambda - c.$$

The next step is to show that

$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 > 0.$$

By direct substitution for p and q, it can be shown that

$$D = \frac{3(3\Delta)^2}{(432\lambda^2\Delta^3)^2}\left\{2\lambda^3 + (1+\lambda)\left(31\lambda^2 + 18\lambda\Delta(1+\lambda) + 27\Delta^2(1+\lambda^2)\right)\right\},$$

which is always positive for all λ and Δ, or $\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3 > 0$.

Hence $D > 0$ is satisfied, and the equation has one real root and two complex roots.

Our final step is to show that this real root is positive by showing

$$u + v = \left(\frac{-q}{2} + \sqrt{D}\right)^{1/3} + \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3} > 0.$$

Let

$$u = \left(\frac{-q}{2} + \sqrt{D}\right)^{1/3} \text{ and } v = \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3}.$$

If $(u+v) > 0$, then $(u+v)^3 > 0$. We have

$$\begin{aligned}(u+v)^3 &= u^3 + v^3 + 3uv(u+v) \\ &= -q - p(u+v)\end{aligned}$$

which is always positive since $p, q < 0$ and $u > v$.

Therefore, we conclude that the equation

$$8(1+\lambda-c)\kappa^3-8\kappa^2-(1+\frac{1}{\lambda})\kappa-(1+\frac{1}{\lambda^2})=0$$

has only one positive real root, given by

$$\left(\frac{-q}{2}+\sqrt{D}\right)^{1/3}+\left(\frac{-q}{2}-\sqrt{D}\right)^{1/3}-\left(\frac{a_1}{3a_0}\right).$$

4. Simulation results

Computer programs were written using SPLUS language to carry out the simulation study on the accuracy of concentration parameter κ obtain by using the above method. Let s be the number of simulations and the following computation were carried out from the simulation study.

i) Mean, $\bar{\hat{\kappa}}=\frac{1}{s}\sum\hat{\kappa}_j$,

ii) Estimated Bias = $\bar{\hat{\kappa}}-\kappa$,

iii) Absolute Relative Estimated Bias (%) = $\left(\frac{|\bar{\hat{\kappa}}-\kappa|}{\kappa}\right)\times 100\%$,

iv) Estimated Standard Errors = $\sqrt{\frac{1}{s-1}\sum\left(\hat{\kappa}_j-\bar{\hat{\kappa}}\right)^2}$,

v) Estimated Root Mean Square Errors (RMSE) = $\sqrt{\frac{1}{s}\sum\left(\hat{\kappa}_j-\kappa\right)^2}$.

The simulation results with $s=200$ and sample size of 100 for each set of true parameter value of κ and λ are shown in Table 1.

	$\kappa=2,\lambda=1$	$\kappa=2,\lambda=1.5$	$\kappa=3,\lambda=2$	$\kappa=4,\lambda=2$
Mean	2.00624	1.98809	2.96821	3.98671
Est Bias	0.00623	-0.01190	-0.03178	-0.01328
Abs Rel Est Bias (%)	0.31197	0.59513	1.05953	0.33215
Est S.E	0.15838	0.15696	0.29359	0.36660
Est RMSE	0.02500	0.02465	0.08677	0.13390

Table 1: Simulation Results for different value of κ and λ.

It is appears that the estimate bias and the absolute relative bias are very small as well as the estimate standard error and estimate root mean square error for all value of parameters. This suggests that the estimate obtained by using this technique is very close the true parameter value.

5. Conclusion

This research has shown that by using the proposed technique we can find the estimate of κ for any value of λ, as opposed to the approximation given by Dobson [2] and Best & Fisher [3] which can only be used when the ratio of error concentration parameter is equal to one or equal error concentration parameter. The estimate obtained is very accurate based on the results of the simulation study.

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