Assessment of secondary students’ mathematical competencies

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Introduction

There is an urgent need to equip young learners with the necessary skills and knowledge in mathematics for the future. In mathematics education, mathematical modeling has increasingly becoming more relevant in developing students' functional competencies required in the 21st century. Students not only need to acquire functional skills such as problem solving and reasoning but the more important question is how to use them in life (OECD, 2009). Mathematical modeling provides a platform where this knowledge can be utilised in building a model to solve real-world problems. This approach is very valuable in the teaching and learning of mathematics (Blum, 2011).

This paper describes the assessment on the modeling competencies of two groups of Form 4 (16 year old) students in Malaysia. The analysis covered the mathematical competencies of identifying variables, making assumptions, mathematics reasoning and interpreting solutions. In addition, this study also reported the challenges and the difficulties students faced when solving the modeling task.

Literature review

Mathematical modeling is the process of translating a real-life problem into a mathematical problem (Ang, 2001). Solving the modeling problem might not be simple as it usually involves integration of a few mathematical concepts (Ang, 2009). According to Blum (2011), mathematical modeling consists of tasks that require the translation between reality and mathematics. Mathematical modeling allows students to experience mathematical situations in real life (Pollak, 1979). A real modeling task would change a person’s view on mathematics as a precise and accurate field to understanding it as having imprecise estimations in reality.

A simple mathematical modeling process consists of four modeling stages, namely, *Observation, Analysis, Interpretation and Application* (Ang, 2001; Swetz & Hartzler, 1991) although the terms used may differ according to researchers. Any modeling process begins with the real world problem that can be formulated into a mathematical problem. The mathematical solutions obtained are usually interpreted in the real-world context before they can be accepted.

One of the important goals of mathematical education is the development of students' mathematical modeling competencies (Chan, Ng, Widjaja & Cynthia 2012). Such development depends on the modeling perspective and the goals intended to be achieved. Most definitions of mathematical competencies involve mathematising the problem and formulating models during the modeling process. The Program for International Students Assessment (PISA) (OECD, 2010) regards modeling competency as a part of mathematical literacy. For a student to be proficient in mathematics, it is not necessary for students to go through every stage of the modeling process.

According to Niss, Blum, and Galbraith (2007), mathematical modeling competency requires the ability to identify the variables, make suitable assumptions, mathematising the real-world problem and interpreting and validating the solution. However, Maass (2006) defined modeling competencies as "skills and abilities to perform modeling process appropriately and are goal oriented as well as willingness to put these into action" (p.117). Jansen (2006) described it as a person's readiness to do something in response when given a mathematical challenge in a situation. Assessment of mathematical modeling competency can be done using the multidimensional approach (Jensen, 2006). This multidimensional paradigm consists of degree of coverage, radius of action and technical level.

Methodology

Sample and location

This study was carried out in a private secondary school in Malaysia. Fifteen students from a mixed-ability class in Form Four 4 (Grade 10) were involved in the research. They worked in groups of three or four. The students had some experience with modeling tasks as the modeling lesson was taught by their mathematics teacher.

Modeling Task

The modeling task was designed by the authors based on the 7-step modeling process by Galbraith (1989, 1995) Figure 1). These steps are an elaboration of the simple mathematical modeling process. In this modeling task, students were required to estimate the maximum height of a building that can be reached by the fire engine ladder. This task was also piloted with a few students to reveal their understanding of the questions in the modeling task. The teacher also went through the whole modeling task with several colleagues to gather feedback that can be used to refine the task.

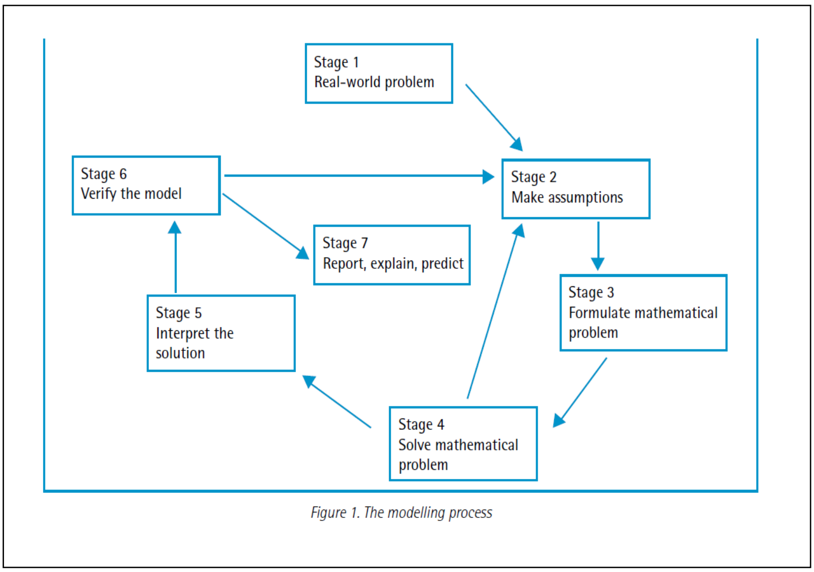

Figure 1. Mathematical modeling cycle process by Galbraith (1989, 1995)

Data collection and analysis

This is an exploratory qualitative analysis of students' responses based on the modeling task developed by the researchers. The students worked in groups to solve the task. In addition, students' responses were also analysed to assess their modeling competencies.

Assessment of students' mathematical competencies

In developing the competencies criteria, several dimensions of modeling competencies mainly from the modeling process by Galbraith (1995) were taken into consideration. In addition, the exposure of these modeling tasks to students for only a short period and students working in groups were also considered. This study focused on the elements of mathematical competencies such as understanding, simplifying, formulating, solving and verifying. The three modeling competencies of making assumptions, computing and interpreting solution and mathematical reasoning are assessed using the rubric developed. This rubric rates the students using the 4-point scale from unsatisfactory to distinguished as shown in Table 1 (next page). This section discusses the assessment of two groups of students concerning their mathematical modeling competencies using the assessment rubrics in Table 1. The assessment was carried out by investigating students' written works and obtaining their responses through interviews.

Exemplification of Band 2 mathematical modeling competencies (Group A)

Group A students were assessed to be in band 2 as they used the stability of a ladder to find the maximum height of a building that can be reached with the fire engine ladder. There are several aspects that show this group of students had difficulties in managing real-world problems from the data collected.

<u>Competence in making assumptions</u>

These students made the assumptions that the farther the distance between the fire engine's ladder and the building the more stable the ladder (see Figure 4). This shows that the students took the safety of the firemen into consideration when finding the maximum height of the building. Although they considered the stability of the ladder, they did not consider its length and the limited area for a fire fighting car to park. The second assumption was the farther the distance between the fire engine's ladder and the building the higher we get to the building. This would place the ladder nearer to the highest point of the building. This assumption is wrong.

<u>Competence in computing and interpreting of solution</u>

Generally, the students had difficulties in understanding the problem statement given; they could not list out the important keywords from the statement or even restate the problem. These difficulties led to their misinterpretation of terms and inability to solve the problem. Based on the responses, the students believed the problem was to estimate the height of the building for the fire engine's ladder to reach it but they overlooked the point about the length of the ladder. The students did not elaborate on the stability and material of the ladder. When the group was discussing, no one brought up the issue of what they must look for to maintain the stability of the ladder. For instance, the angle of elevation of the ladder could be the factor. In this case the students were unable to make an assumption based on the problem situation involving the length of the ladder and the height of the building.

Table 1. Rubric for Assessing Mathematical Modeling Competencies

Competencies	Band 1 (Unsatisfactory)	Band 2 (Basic)	Band 3 (Proficient)	Band 4 (Distinguished)
Making assumption	▪ No keywords listed. ▪ No variables listed. ▪ No assumptions made. ▪ Incorrect notions of assumptions.	▪ Less than 2 variables listed or assumptions made based on real-world interpretations of task. ▪ Assumptions stated are irrelevant to model.	▪ More than 2 variables or assumptions made based on real-world interpretations of task. ▪ Assumptions stated are relevant to model.	▪ Comprehensive list of variables or assumptions made based on real-world interpretations of task. ▪ Assumptions stated are relevant to model.
Computing and interpreting solution	▪ Never show attempt to develop mathematical model. ▪ Errors shown in computation. ▪ There is no evidence of real-world constraint in the presentation of work.	▪ Show little attempt to develop mathematical model. ▪ Minor errors shown in computation. ▪ Show evidence of 1 real- world consideration in examining variables that will impact interpretation and solution of modeling task.	▪ Attempt to develop a proper mathematical model. ▪ No errors shown in computation. ▪ Show evidences of 2 real- world consideration in examining variables that will impact interpretation and solution of modeling task.	▪ Develop a perfect mathematical model. ▪ Computation is clear and accurate. ▪ Show evidences of more than 2 real-world consideration in examining variables that will impact interpretation and solution of modeling task.
Mathematical reasoning	▪ Mathematical reasoning is not logical. ▪ Inappropriate use of mathematics.	▪ Mathematical reasoning is somewhat logical. ▪ Inappropriate use of mathematics.	▪ Mathematical reasoning is logical. ▪ Appropriate use of mathematics.	▪ Mathematical reasoning is logical. ▪ Correct use of mathematics.

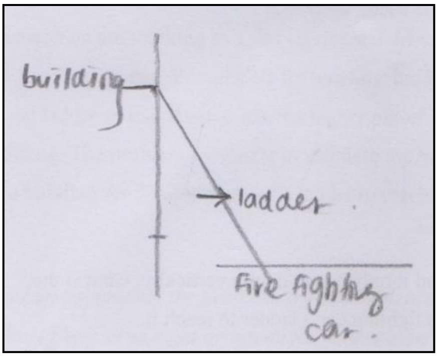

Figure 2. Mathematical model from Group A

Students attempted to draw a mathematical model of the real world situation using a direction arrow (see Figure 2). But there is an error in their model because the arrow cannot be explained using a mathematical concept. After interviewing this group of students, it was discovered that the arrow on the ladder means, "as we increase the horizontal distance between the fire engine and the building, the more stable the ladder is." In the end, the students in Group A were unsuccessful in finding the solution to the modeling task.

Competence in mathematical reasoning

The students in Group A made wrong assumptions and formulated the wrong model. This resulted in their inability to explain the reason behind their solution. There is a possibility that students were unable to use the mathematics concepts and this caused a wrong interpretation. This led to students' inability to give a good solution or values for the mathematical modeling task. In addition, the students were struggling to give a clear explanation. One student shared that the answer to this problem could only be found if the length of the ladder were provided.

Exemplification of Band 3 mathematical modeling competencies (Group B)

Group B was assessed to be at band 3 of mathematical modeling competence. They showed more understanding of the problem and more mathematizing effort. The students managed to list the important keywords from the problem statement and were able to make appropriate assumptions. They identified the important variables from the problem and formulated mathematical relationships for these. The mathematical reasoning of Group B was more logical and accurate.

Competence in making assumptions

The students in Group B listed the three variables affecting the estimation of the maximum height of the building from the fire engine's ladder. The variables were the distance between the building and the fire engine, angle of elevation for the ladder and the height of the fire engine. These variables indicated that the students understood the importance of listing the variables. This possibly shows their understanding of the task.

Although Group B students discussed the length of the ladder as a variable, they did not write it down in the making assumptions section. They assumed that the length of the ladder is 30 m and the distance between the building and the fire engine is about 1m to 3 m. They were unable to find more information on the typical length of the ladder in a fire engine. The value the students used was more for testing their model. Nonetheless, the value was suitable to test the assumptions. In addition, they estimated the height of the fire fighting car to be 3 meters to 4 meters and the best angle of elevation for the ladder to be 45°.

Competence in computing and interpreting of solution

Group B students sketched an appropriate mathematical model in stage 3 and used the value written in stage 2. From the diagram (see Figure 3), the horizontal rectangle represented the fire engine while the vertical rectangle represented the building. There was an angle labeled as 45^{0} degrees. The distance between the building and the engine was 1.5 m. This model fulfilled all the variables that they had written in stage 2 and can be considered as a geometric mathematical model.

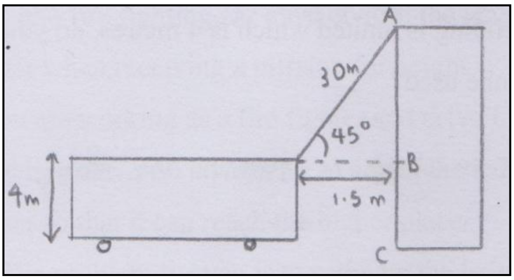

Figure 3. Mathematical model from Group B

The Group B students were able to formulate the fire engine problem into a mathematical problem. Their solutions are presented in the next section.

Competence in mathematical reasoning

The students calculated the maximum height of a building that can be reached using the fire engine ladder by using their previous knowledge of trigonometry. They used the basic trigonometry sine 45° to find the height of the building and the height of the fire engine. The sum of these two values gives the maximum height of the building (see Figure 4).

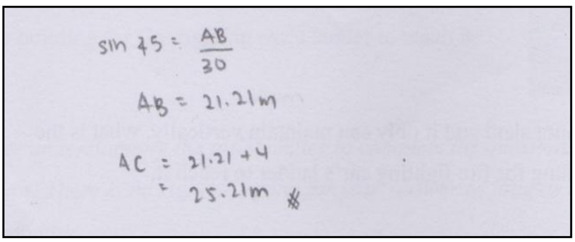

Figure 4. Solution from Group B

The students got 25.12 m from their calculation of the maximum height of the building. The written response showed that they were able to explain their solution and reason out why these values had been used. The mathematical model from Group B students was correct. The solution was not accurate because they labeled 1.5 m as their distance between the fire fighting car and the building and other factors were not taken into account. Hence, they were unable to consider the limitation due to the numeric values chosen. Nonetheless, they went on to complete the task correctly in stage 5. From here, the students showed appropriate use of mathematics but the values used in the model were inaccurate.

Discussion and conclusion

The modeling competencies of both groups showed weaknesses in making assumptions, computing and interpreting solutions and mathematical reasoning. The students in Group A were assessed to be basic users (Band 2) while Group B students performed slightly better to be proficient users (Band 3). Group A students struggled in making assumptions and formulating a model and had some difficulties in mathematical reasoning. However, Group B students performed slightly better in making assumptions and formulating a model compared to Group A students. Furthermore, the students from both groups found making generalisations on why their model works to be very challenging. This result is similar to the findings by Chan, Ng, Widjaja and Cynthia (2012).

The development of modeling competencies requires regular exposure to tasks over time. It is a continuous process that should be provided to students to enhance their understanding of mathematical concepts, which is a next step in acquiring the modeling competencies. This would require a longer duration for best effect (Lesh & Doerr, 2003). Teachers should also familiarise themselves with the modeling tasks in the beginning. According to Maass (2006), teachers need to know suitable pedagogical methods when solving modeling tasks to support the development of students' modeling competencies in the classroom.

In conclusion, this study has shown that beginner modellers are capable of solving a modeling task at different levels of competence. With more engagement in the modeling tasks, it is hoped that students would be able to acquire modeling abilities and develop their modeling competencies. More studies should be done to assess other modeling competencies that are important in the modeling process.

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Appendix

Fire Engine Ladder

The driver of a fire engine must control the length of the ladder when receiving an emergency call for rescue. You are working as a fire fighter and one of your job[s] is to drive the fire engine. It is important to adjust the length of the ladder so that it reaches the highest section of the building. In this situation, estimate the maximum height of the building that can be reached by a fire engine ladder.

After understanding the problem, try to complete the questions/ instruction in the box below. There is no right or wrong answer [for] as long as it is reasonable and acceptable.

<u>Stage 1</u>

i. List out keywords.

ii. Restating Problem.

<u>Stage 2</u>

i. List out the variables that will influence the estimation of the maximum height of a building that can be reached by a fire engine's ladder

ii. How do you make assumptions on these variables? And more importantly, how do you justify the assumptions you have made?

<u>Stage 3</u>

You are now [be] given a chance to formulate the problem into a mathematical model.

<u>Stage 4</u>

Solve the mathematical problem that you had formulated and interpret your solution based on the problem given. (Explain why.)

<u>Stage 5</u>

Make a prediction for the following situation based on your solution.

i. If the space around the building is limited which is 4 metres, can your estimation/ results still be used?

ii. If the ladder cannot slant and it only can maintain vertically, what is the maximum height of a building using the ladder from the fire engine?