Bounded distance preserving surjective mappings on block triangular matrix algebras

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Abstract

Let $\mathcal{M}_n$ be the algebra of $n \times n$ square matrices. Let $\mathcal{T}$ and $\mathcal{U}$ be block triangular matrix subalgebras of $\mathcal{M}_n$ and $\mathcal{M}_m$, respectively. Let r be an integer such that $1 \leqslant r < \min\left\{\lceil \frac{n+1}{2} \rceil, \lceil \frac{m+1}{2} \rceil\right\}$. In this talk, we show that every surjective mappings $\psi : \mathcal{T} \to \mathcal{U}$ satisfying

$$\operatorname{rank}(A - B) \le r \quad \Leftrightarrow \quad \operatorname{rank}(\psi(A) - \psi(B)) \le r$$

are bijective mappings preserving adjacency in both directions.

Keywords: Block triangular matrix, adjacency preserving mapping, bounded distance preserving mapping, geometry of matrices, rank.

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Bounded distance preserving surjective mappings on block triangular matrices

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Introduction

Related to the theory of functions of several complex variables and Siegel's geometry, **Professor L.K. Hua** initiated the study of *geometry of matrices* in the middle of forties of the last century.

Introduction

In the geometry of matrices, the *points* of the associated matrix space $\mathcal{M}$ are a certain kind of matrices of a given size, and the *arithmetic distance*, or simply the *distance*, of two points of $\mathcal{M}$ is the rank of their difference.

Two points of $\mathcal{M}$ are said to be *adjacent* if their distance is one or minimal.

Hua discovered that the invariant of **adjacency** alone is sufficient to characterize the transformation groups of the geometries of four types of matrices, namely rectangular matrices, symmetric matrices, Hermitian matrices and alternate matrices.

Introduction

In the fundamental theorem of the geometry of matrices, every bijective mapping ψ on the associated matrix space for which ψ and ψ^{-1} preserve adjacency (or equivalently, bijective mapping ψ preserving adjacency in both directions) is characterized.

For an extensive expository survey of the geometry of matrices, see the books of Wan (1996), and the references therein.

- Z.X. Wan, *Geometry of matrices*, World Scientific, Singapore, 1996.

Introduction

Inspired by the fundamental results of Hua's work in this area, it has been attracted and followed by many mathematicians.

- similar problems have been studied in various of finite and infinite dimensional linear spaces;
- the fundamental theorem of the geometry of matrices has been improved under weaker assumptions:
  - dropping *bijectivity* assumption,
    - P. Šemrl, *On Hua's fundamental theorem of the goemetry of rectangular matrices.* Journal of Algebra. **248** (2002) 366-380.
  - replacing *preserving adjacency in both directions* by *preserving adjacency*(in one direction);
    - P. Šemrl, *Hua's fundamental theorem of the geometry of matrices.* Journal of Algebra. **272** (2004) 801-837.
    - W.L. Huang, Z.X. Wan, *Adjacency preserving mappings on rectangular matrices.* Beitr. Algebra Geom. **45 (2)** (2004) 435-446.
    - W.L. Huang, R. Hofer, Z.X. Wan *Adjacency preserving mappings of symmetric and Hermitian matrices.* Aequationes Math. **67** (2004) 132-139.

Introduction

- some continuation studies have been considered in classifying surjective mappings preserving pairs of elements with maximal distance, bounded distance, as well as, fixed distance in both directions.

Motivation

In 2006, motived by a result concerning geometric mappings in Grassmann spaces, Havlicek and Šemrl, using Hua's fundamental theorem, classified bijective mappings ψ preserving maximal distance in both directions (i.e., $A-B$ is of full rank $\Leftrightarrow$ $\psi(A)-\psi(B)$ is of full rank) on rectangular matrices.

- H. Havlicek, P. Šemrl, *From geometry to invertibility preservers.* Studia Math. **174** (2006) 99-109.

Motivation

Later on, Lim and Tan characterized surjective mappings ψ on the spaces of rectangular matrices, symmetric matrices, Hermitian matrices and alternate matrices preserving bounded distance in both directions, namely

$$\rho(A - B) \leqslant r \quad \Leftrightarrow \quad \rho(\psi(A) - \psi(B)) \leqslant r$$

for some integer $1 < r < \ell$, where ρ is the rank function and ℓ is the maximal rank of matrices of the space.

- M.H. Lim, J.J.H. Tan, *Preservers of matrix pairs with bounded distance.* Linear Algebra Appl. **422** (2007) 517-525.
- M.H. Lim, J.J.H. Tan, *Preservers of pairs of bivectors with bounded distance.* Linear Algebra Appl. **430** (2009) 564-573.

Motivation

In 2008, Huang and Havlicek, with a different approach, by considering a *class of graphs* subjects to *five conditions* including the finiteness of diameters, showed that the maximal distance preservation in both directions implies the adjacency preservation in both directions, and then, this result was applied to the graphs arising from the adjacency relations for four kinds of matrix spaces, namely the spaces of rectangular matrices, symmetric matrices, Hermitian matrices and alternate matrices, as well as for the Grassmann spaces.

- W.L. Huang, H. Havlicek, *Diameter preserving surjections in the geometry of matrices*, Linear Algebra Appl. **429** (2008) 376-386.

Motivation

By a similar idea employed in the paper of Huang and Havlicek, Huang studied surjective mappings preserving bounded distance in both directions on the spaces of rectangular matrices, symmetric matrices, Hermitian matrices and alternate matrices, Grassmann spaces, and classical dual polar spaces in the following two papers

- W.L. Huang, *Bounded distance preserving surjections in the geometry of matrices*, Linear Algebra Appl. **433** (2010) 1973-1987.
- W.L. Huang, *Bounded distance preserving surjections in the projective geometry of matrices*, Linear Algebra Appl. **435** (2011) 175-185.

Motivation

In 2010, L.P. Huang, in his paper

- L.P. Huang, *Good distance graphs and the geometry of matrices*, Linear Algebra Appl. **433** (2010) 221-232,

introduced the concept of a good distance graph and proved the following assertions are equivalent

- ψ is a graph isomorphism.
- ψ is a surjective mapping preserving a fixed bounded distance k in both directions.
- ψ is a surjective mapping preserving a fixed distance $k/2$ in both directions.

Then, he characterized surjective mappings preserving bounded distance as well as fixed distance on Hermitian matrices and symmetric matrices.

Geometry of Block Triangular Matrices

and

Surjective Mappings Preserving Bounded Distance in Both Directions

Geometry of Block Triangular Matrices

Motivated by Hua's pioneer work in the geometry of matrices, in 2002, Chooi and Lim initiated the study of the geometry of block triangular matrices over an arbitrary field in the following paper

- W.L. Chooi, M.H. Lim, *Coherence invariant mappings on block triangular matrix spaces*, Linear Algebra Appl., **346** (2002) 199-238.

They characterized bijective mappings preserving adjacency in both directions on block triangular matrices. The result is quite different from and more complicated that the corresponding theorem on spaces rectangular matrices, symmetric matrices, Hermitian matrices and alternate matrices.

Bounded Distance Preserving Mappings

As a continuation work, we now study surjective mappings preserving bounded distance in both directions on block triangular matrices.

For the sake of our discussion, we need some notation.

- $\mathcal{M}_n$ is the space of n-square matrices over a field $\mathbb{F}$.
- $\mathcal{T}_{n_i,k}$ denotes a subalgebras of $\mathcal{M}_n$ consisting of $k \times k$ block upper triangular matrices (A_{ij}) with $A_{ii} \in \mathcal{M}_{n_i}$ for $i = 1, \ldots, k$ and $n_i + \cdots + n_k = n$.
- E_{ij} stands for the matrix unit whose (i,j)-th entry is one and the others are zero.

For each block triangular matrix subalgebra $\mathcal{T}_{n_i,k}$ of $\mathcal{M}_n$, we denote by

- $\max(n)$ the greatest integer $1 \leqslant j \leqslant n$ such that $E_{j,n+1-j} \in \mathcal{T}_{n_i,k}$.

Main Theorem

Theorem 1

Let $\mathcal{T}_{n_i,k}$ and $\mathcal{T}_{m_i,h}$ be block triangular matrix subalgebras of $\mathcal{M}_n$ and $\mathcal{M}_m$, respectively, over a field with at least three elements, and $n_1, n_k = 1$ or $n_1, n_k \geqslant 2$. Let r be an integer such that $1 \leqslant r < \min\{\max(m), \max(n)\}$. Let $\psi : \mathcal{T}_{n_i,k} \to \mathcal{T}_{m_i,h}$ be a surjective mapping satisfying

$$\rho(A-B) \leqslant r \quad \Leftrightarrow \quad \rho(\psi(A)-\psi(B)) \leqslant r \tag{1}$$

for every $A, B \in \mathcal{T}_{n_i,k}$. Then ψ is a bijective mapping preserving adjacency in both directions, and $\mathcal{T}_{m_i,h} = \mathcal{T}_{n_i,k}$ or $\mathcal{T}_{m_i,h} = \mathcal{T}_{n_{k+1-i},k}$.

Examples

Example 1

Let k and n be positive integers such that $k \geqslant 2$ and $n \geqslant 3$. Let $\mathcal{T}_{n_i,k}$ be a block triangular matrix algebra such that $n_i = n_j$ for every $1 \leqslant i,j \leqslant k$. Let σ be a permutation of degree k, and let $\psi : \mathcal{T}_{n_i,k} \to \mathcal{T}_{n_i,k}$ be the mapping defined by

$$\psi(A) = \begin{pmatrix} A_{\sigma(1)\sigma(1)} & A_{12} & \cdots & A_{1k} \\ 0 & A_{\sigma(2)\sigma(2)} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{\sigma(k)\sigma(k)} \end{pmatrix}$$

for every block triangular matrix $A = (A_{ij}) \in \mathcal{T}_{n_i,k}$ with $A_{ij} \in \mathcal{M}_{n_i,n_j}$ for $1 \leqslant i \leqslant j \leqslant k$. It is easily checked that ψ is a bijective mapping satisfying satisfying condition (1) for $r = n$, but it does not preserve adjacency in both directions.

Examples

Example 2

Let $\mathcal{T}_5$ denote the algebra of 5×5 upper triangular matrices over a field. Let $\psi : \mathcal{T}_5 \to \mathcal{T}_5$ be the mapping defined by

$$\psi(A) = A + (a_{24} - a_{15})E_{15} + (a_{15} - a_{24})E_{24}$$

for every $A = (a_{ij}) \in \mathcal{T}_5$. We note that ψ is a bijective mapping satisfying condition (1) for $r = 4, 5$. But, ψ does not preserve adjacency in both directions.

Key Lemmas

Let $1 \leqslant r \leqslant n$ and let $\mathcal{T}_{n_i,k}$ be a block triangular matrix subalgebra of $\mathcal{M}_n$. For each nonempty subset S of $\mathcal{T}_{n_i,k}$, we define

$$S^{\perp_r} := \{\, T \in \mathcal{T}_{n_i,k} \mid \rho(T - X) \leqslant r \ \text{ for all } X \in S\}$$

and

$$S^{\perp_r\perp_r} := (S^{\perp_r})^{\perp_r}.$$

Lemma 1

Let $\mathcal{T}_{n_i,k}$ be a block triangular matrix subalgebra of $\mathcal{M}_n$ over a field $\mathbb{F}$ with at least three elements. Let $1 \leqslant r < \max(n)$. If $A, B \in \mathcal{T}_{n_i,k}$ with $\rho(B - A) = 1$, then $\left|\{A, B\}^{\perp_r\perp_r}\right| \geqslant 3$.

Key Lemmas

Lemma 2

Let $A, B \in \mathcal{T}_{n_i,k}$ *be matrices with* $1 \leqslant \rho(B-A) = h \leqslant r < \max(n)$ *and*

$$B - A = P(E_{s_1t_1} + \cdots + E_{s_ht_h})Q$$

for some invertible matrices $P, Q \in \mathcal{T}_{n_i,k}$, *where* $E_{s_1t_1}, \ldots, E_{s_ht_h}$ *in* $\mathcal{T}_{n_i,k}$ *with* $1 \leqslant s_j \leqslant t_j \leqslant n$ *for* $j = 1, \ldots, h$, $1 \leqslant s_1 < \cdots < s_h \leqslant n$, *and* $1 \leqslant t_i \neq t_j \leqslant n$ *for every* $1 \leqslant i \neq j \leqslant h$. *If one of the following conditions hold:*

- $h = 2$, *and* $(n_1, s_1, t_1) \neq (1, 1, 1)$ *and* $(n_k, s_2, t_2) \neq (1, n, n)$,
- $h = 3$, *and* $(n_1, s_1, t_1) \neq (1, 1, 1)$ *and* $(n_k, s_3, t_3) \neq (1, n, n)$,
- $h \geqslant 4$,

then $\{A, B\}^{\perp_r \perp_r} = \{A, B\}$.

Application

By a similar technique used in the proof of L.P. Huang's paper

- L.P. Huang, *Good distance graphs and the geometry of matrices*, Linear Algebra Appl. **433** (2010) 221-232,

together with Theorem 1, we obtain

Theorem 2

Let $\mathcal{T}_{n_i,k}$ and $\mathcal{T}_{m_i,h}$ be block triangular matrix subalgebras of $\mathcal{M}_n$ and $\mathcal{M}_m$, respectively, over a field $\mathbb{F}$ with at least three elements, and $n_1, n_k = 1$ or $n_1, n_k \geqslant 2$. Let s be an integer such that $1 \leqslant s < \min\left\{\frac{\max(m)}{2}, \frac{\max(n)}{2}\right\}$. Let $\psi : \mathcal{T}_{n_i,k} \to \mathcal{T}_{m_i,h}$ be a surjective mapping satisfying

$$\rho(A - B) = s \quad \Leftrightarrow \quad \rho(\psi(A) - \psi(B)) = s$$

for every $A, B \in \mathcal{T}_{n_i,k}$. Then ψ is a bijective mapping preserving adjacency in both directions, and $\mathcal{T}_{m_i,h} = \mathcal{T}_{n_i,k}$ or $\mathcal{T}_{m_i,h} = \mathcal{T}_{n_{k+1-i},k}$.