On a Class of Real Hypersurfaces in a Complex Space Form

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(Paper presented at the ***The Conference on Geometry - Theory and Applications*** held on 20-24 Jun 2011: Styria, Austria)

manuscript No.
(will be inserted by the editor)

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Received: date / Accepted: date

Abstract Let M be a real hypersurface in a complex space form $M_n(c)$, A the shape operator and ϕ the almost contact structure on M induced by the complex structure on $M_n(c)$. In this paper, we study the condition

$$\langle(\phi A - A\phi)X, Y\rangle = 0$$

for X, Y in $\Gamma(\mathscr{D})$, where $\mathscr{D}$ is the holomorphic distribution on M. In particular, we classify minimal real hypersurfaces satisfying this condition in complex Euclidean spaces and in complex projective spaces.

Keywords Complex Euclidean spaces · complex projective spaces · Hopf hypersurfaces · ruled real hypersurfaces

Mathematics Subject Classification (2000) 53B25 · 53C15

1 Introduction

Let $M_n(c)$ be an n-dimensional complete and simply connected complex space form with complex structure J of constant holomorphic sectional curvature $4c$, i.e., it is either a complex projective space $\mathbb{C}P^n$ (for $c > 0$), a complex Euclidean space $\mathbb{C}^n$ (for $c = 0$) or a complex hyperbolic space $\mathbb{C}H^n$ (for $c < 0$).

This work was supported in part by the FRGS research grant (Grant No. FP059/2008C).

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A connected real hypersurface M in $M_n(c)$ is said to be *Hopf* if the structure vector field ξ is principal, i.e., $A\xi = \alpha\xi$, for some function α on M, where $\xi = -JN$ and N a unit vector normal to M (cf. [3]).

In 1973, Takagi [28] classified homogeneous real hypersurfaces in $\mathbb{C}P^n$ into six classes of Hopf hypersurfaces with constant principal curvatures, nowaday, known as real hypersurfaces of type A_1, A_2, B, C, D and E. Cecil and Ryan [4] showed that Hopf hypersurfaces could be expressed as tubes of constant radius over certain complex submanifolds in the ambient space. Having extended the results of Cecil and Ryan in [4], Kimura [14] showed that the Hopf hypersurfaces in $\mathbb{C}P^n$ with constant principal curvatures are indeed the homogeneous real hypersurfaces.

On the other hand, Montiel [23] obtained a list of homogeneous Hopf hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures, nowaday so-called of type A_0, A_1, A_2 and B. Berndt [3] showed that the real hypersurfaces in the Montiel's list are the only Hopf hypersurfaces in $\mathbb{C}H^n$ with constant principal curvatures.

A typical example of non-Hopf real hypersurfaces in $M_n(c)$, for $c \neq 0$, is the class of ruled real hypersurfaces. *Ruled real hypersurfaces* in $M_n(c)$ are characterized by having a one-codimensional foliation whose leaves are totally geodesic complex hypersurfaces in $M_n(c)$ (cf. [21]).

These real hypersurfaces mentioned above appeared to be standard spaces and play a central role in the study of real hypersurfaces in a non-flat complex space form. In the past two decades, a number of papers dealt with the problem of characterizing real hypersurfaces in a non-flat complex space form under certain additional properties on which the real hypersurfaces being classified consist of subclasses of the Takagi's list, Montiel's list and ruled real hypersurfaces (see [10], [12], [16]-[20], etc, for some recent papers and also the papers cited in [25]).

Real hypersurfaces, other than these standard spaces, have little been investigated, partly because the properties studied are too restrictive to be used for characterizing other classes of real hypersurfaces. It is natural to investigate certain conditions weak enough to include other classes of real hypersurfaces into the classification. This paper is a contribution along this line.

We shall now briefly discuss the motivation for the paper as well as the results obtained.

Besides the submanifold structure, represented by the shape operator A, on M, there is an almost contact metric structure $(\phi, \xi, \eta, \langle,\rangle)$ on M induced by the complex structure J of the ambient space. In [24], [27] and [26], having considered the condition

$$\phi A = A\phi \tag{1}$$

the authors then investigated the interaction of these two structures. From the Sasakian geometrical viewpoint, the condition (1) is necessary and sufficient for the the almost contact structure on M to be normal.

We combine their results in the following theorem.

Theorem 1 ([24], [27], [26]) *Let M be a real hypersurface in $M_n(c)$, $n \geq 2$. Then M satisfies (1) if and only if M is locally congruent to one of the following spaces:*

1. For $c > 0$
  (A_1) geodesic spheres,
  (A_2) tubes over a totally geodesic $\mathbb{C}P^p$, for $p \in \{1, 2, \cdots, n-2\}$.
2. For $c = 0$
  (a) totally geodesic hyperplanes $\mathbb{R}^{2n-1}$,

(b) *hyperspheres* $S^{2n-1}(a)$, $a>0$,
(c) *the product spaces* $S^{2p-1}(a)\times\mathbb{C}^{n-p}$, $a>0$,
(d) *cylinders over a plane curve.*

3. *For* $c<0$
(A_0) *horospheres,*
(A_1) *geodesic hyperspheres and tubes over totally geodesic complex hyperplanes* $\mathbb{C}H^{n-1}$,
(A_2) *tubes over totally geodesic* $\mathbb{C}H^p$, *for* $p\in\{1,2,\cdots,n-2\}$.

The real hypersurfaces M listed in this theorem are Hopf. For $c\neq 0$, the real hypersurfaces are non-ruled, while for $c=0$, the real hypersurfaces are ruled for cases (a), (d) and in (c) provided that $p=1$.

This theorem completely classify real hypersurfaces satisfying condition (1). It is interesting to look at a condition that is weaker than (1). The main objective of this paper is to study the condition

$$\langle(\phi A-A\phi)X,Y\rangle=0 \tag{2}$$

for any $X,Y\in\Gamma(\mathscr{D})$, where $\mathscr{D}:=\mathrm{Span}\{\xi\}^{\perp}$ is the *holomorphic distribution* on M and $\Gamma(\mathscr{V})$ denotes the module of all differentiable sections on a vector bundle $\mathscr{V}$ over M.

We first prove the following result in Sect. 3.

Theorem 2 *Let M be a real hypersurface in $M_n(c)$, $n\geq 3$. Suppose that M satisfies the condition*

$$d\alpha(\xi) \text{ is nowhere zero in an open dense subset of } M, \tag{$*$}$$

where $\alpha=\langle A\xi,\xi\rangle$. If M satisfies (2) then M is locally congruent to a ruled real hypersurface.

It is worthwhile to remark that there exist many examples of ruled real hypersurfaces M in $M_n(c)$ on which the condition ($*$) does not hold. For instance, the minimal ruled real hypersurfaces given in [1, 15]. However, there do exist ruled real hypersurfaces in $M_n(c)$ with $d\alpha(\xi)\neq 0$ everywhere (cf. [7]).

Surprisingly, Theorem 2 is not true if the condition (*) is omitted. In Sect. 5, we shall exhibit an example to illustrate this remark. More precisely, we study the immersion

$$\Psi:\mathbb{C}^{p_0}\times\mathbb{R}^+\times S^{2p_1+1}(\hat{a})\times S^{2p_2+1}(\hat{b})\to\mathbb{C}^n=\mathbb{C}^{p_1+1}\times\mathbb{C}^{p_2+1}\times\mathbb{C}^{p_0}$$

given by

$$\Psi(z,r,x_1,x_2)=(rx_1,rx_2,z)$$

where $\hat{a}$, $\hat{b}>0$ with $\hat{a}^2+\hat{b}^2=1$ and p_0, p_1, p_2 three non-negative integers with $p_0+p_1+p_2+2=n$. Ψ immersed $M=\mathbb{C}^{p_0}\times\mathbb{R}^+\times_r\left(S^{2p_1+1}(\hat{a})\times S^{2p_2+1}(\hat{b})\right)$ isometrically into $\mathbb{C}^n$ as a real hypersurface. It can be verified that such a real hypersurface satisfies both (2) and the property $d\alpha(\xi)=0$ everywhere. We can also observed that it is non-ruled unless $p_1=p_2=0$.

In the last two sections, we focus on the case: $d\alpha(\xi)=0$ everywhere. This case seems to be more complicated, a slightly stronger condition is hence, being imposed in our study.

We first consider the complex Euclidean ambient space in Sect. 6. With $V:=\phi A\xi$ and $\beta:=||V||$, we show that under certain restriction, real hypersurfaces in $\mathbb{C}^n$ satisfying (2) are defined by the above immersion, i.e.,

Theorem 3 *Let p_0, p_1, p_2 be three non-negative integers with $p_0+p_1+p_2+2=n$ and $\hat{a}$, $\hat{b}>0$ with $\hat{a}^2+\hat{b}^2=1$. Then the mapping*

$$\Psi:\mathbb{C}^{p_0}\times\mathbb{R}^+\times S^{2p_1+1}(\hat{a})\times S^{2p_2+1}(\hat{b})\to\mathbb{C}^n=\mathbb{C}^{p_1+1}\times\mathbb{C}^{p_2+1}\times\mathbb{C}^{p_0}$$

given by

$$\Psi(z,r,x_1,x_2)=(rx_1,rx_2,z)$$

defines a real hypersurface M with $\operatorname{grad}\alpha=\alpha V$, *non-vanishing β and satisfying the condition (2).*

Conversely, if M is a real hypersurface in $\mathbb{C}^n$, $n\geq 3$, with $\operatorname{grad}\alpha=\alpha V$, *non-vanishing β and satisfying the condition (2), then up to rigid motions of $\mathbb{C}^n$, M is defined by the above immersion Ψ.*

By the above theorem, we obtain the following classification of minmal real hypersurfaces in $\mathbb{C}^n$ satisfying (2).

Theorem 4 *Let p_0, p_1, p_2 be three non-negative integers with $p_0+p_1+p_2+2=n$ and*

$$\hat{a}=\sqrt{\frac{2p_1+1}{2(p_1+p_2+1)}},\quad \hat{b}=\sqrt{\frac{2p_2+1}{2(p_1+p_2+1)}}.$$

Then the mapping

$$\Psi:\mathbb{C}^{p_0}\times\mathbb{R}^+\times S^{2p_1+1}(\hat{a})\times S^{2p_2+1}(\hat{b})\to\mathbb{C}^n=\mathbb{C}^{p_1+1}\times\mathbb{C}^{p_2+1}\times\mathbb{C}^{p_0}$$

given by

$$\Psi(z,r,x_1,x_2)=(rx_1,rx_2,z)$$

defines a minimal real hypersurface M with non-vanishing β and satisfying the condition (2).

Conversely, if M is a minimal real hypersurface in $\mathbb{C}^n$, $n\geq 3$, with non-vanishing β and satisfying the condition (2), then up to rigid motions of $\mathbb{C}^n$, M is defined by the above immersion Ψ.

Finally, in the last section, we deal with complex projective ambient spaces and prove the following result.

Theorem 5 *Let p_0, p_1, p_2 be three non-negative integers with $p_0+p_1+p_2+1=n$ and $\hat{a}$, $\hat{b}>0$ with $\hat{a}^2+\hat{b}^2=1$. Then the mapping*

$$\Psi(z,r,x_1,x_2)=\psi(rx_1,rx_2,z) \tag{3}$$

where $\psi:\mathbb{C}^{n+1}_*\to\mathbb{C}P^n$ is the canonical projection, $z\in\mathbb{C}^{p_0}$, $r\in\mathbb{R}^+$, $x_1\in S^{2p_1+1}(\hat{a})$, $x_2\in S^{2p_2+1}(\hat{b})$, defines a real hypersurface M with $\operatorname{grad}\alpha=\alpha V$ *and satisfying the condition (2).*

Conversely, if M is a real hypersurface in $\mathbb{C}P^n$, $n\geq 3$, with $\operatorname{grad}\alpha=\alpha V$ *and satisfying the condition (2), then up to rigid motions of $\mathbb{C}^{n+1}$, M is defined by the above immersion Ψ.*

We then apply this theorem to classify minimal real hypersurfaces in $\mathbb{C}P^n$ satisfying (2).

Theorem 6 *Let p_0, p_1, p_2 be three non-negative integers with $p_0+p_1+p_2+1=n$ and*

$$\hat{a}=\sqrt{\frac{2p_1+1}{2(p_1+p_2+1)}},\quad \hat{b}=\sqrt{\frac{2p_2+1}{2(p_1+p_2+1)}}.$$

Then the mapping

$$\Psi(z,r,x_1,x_2)=\psi(rx_1,rx_2,z)$$

where $\psi:\mathbb{C}_*^{n+1}\to\mathbb{C}P^n$ is the cannonical projection, $z\in\mathbb{C}^{p_0}$, $r\in\mathbb{R}^+$, $x_1\in S^{2p_1+1}(\hat{a})$, $x_2\in S^{2p_2+1}(\hat{b})$, defines a minimal real hypersurface M satisfying the condition (2).

Conversely, if M is a minimal real hypersurface in $\mathbb{C}P^n$, $n\geq 3$, satisfying the condition (2), then up to rigid motions of $\mathbb{C}^{n+1}$, M is defined by the above immersion Ψ.

2 Preliminaries

In this section we shall recall some fundamental identities in the theory of real hypersurfaces in a complex space form and fix some notations. Some general results are also derived here.

Let M be a connected real hypersurface isometrically immersed in $M_n(c)$, $n\geq 3$, N a unit normal vector field on M and $\langle,\rangle$ the Riemannian metric on M. We define a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η by

$$JX=\phi X+\eta(X)N,\quad JN=-\xi,\quad \eta(X)=\langle\xi,X\rangle$$

for any $X\in\Gamma(TM)$. Then we have

$$\phi^2X=-X+\eta(X)\xi,\quad \phi\xi=0,\quad \eta(\phi X)=0,\quad \eta(\xi)=1. \tag{4}$$

Denote by ∇ the Levi-Civita connection and A the shape operator on M. Then

$$(\nabla_X\phi)Y=\eta(Y)AX-\langle AX,Y\rangle\xi,\quad \nabla_X\xi=\phi AX \tag{5}$$

for any $X,Y\in\Gamma(TM)$.

Let R be the curvature tensor of M. Then the equations of Gauss and Codazzi are given respectively by

$$\begin{aligned}R(X,Y)Z=c\{&\langle Y,Z\rangle X-\langle X,Z\rangle Y+\langle\phi Y,Z\rangle\phi X-\langle\phi X,Z\rangle\phi Y\\&-2\langle\phi X,Y\rangle\phi Z\}+\langle AY,Z\rangle AX-\langle AX,Z\rangle AY\end{aligned}$$

$$(\nabla_XA)Y-(\nabla_YA)X=c\{\eta(X)\phi Y-\eta(Y)\phi X-2\langle\phi X,Y\rangle\xi\}.$$

The second order covariant derivative ∇^2A on the shape operator A is defined by

$$(\nabla^2_{XY}A)Z=\nabla_X\{(\nabla_YA)Z\}-(\nabla_{\nabla_XY}A)Z-(\nabla_YA)\nabla_XZ.$$

The following lemma characterizes ruled real hypersurfaces in $M_n(c)$.

Lemma 1 **([22])** *Let M be a real hypersurface in $M_n(c)$, $n\geq 2$. Then M is a ruled real hypersurface if and only if $\phi A\phi=0$, or equivalently $\langle AX,Y\rangle=0$, for any $X,Y\in\Gamma(\mathscr{D})$.*

In the following, we denote by $V := \phi A\xi$, $\alpha := \eta(A\xi)$ and $\beta := ||V||$. Also, we define a symmetric tensor field T of type (1,1) by

$$TX := (\nabla_\xi A)X - \langle X, V\rangle A\xi - \eta(AX)V \tag{6}$$

for any $X \in \Gamma(TM)$. Further we let

$$G_1 := \{x \in M : ||\phi A\phi|| \neq 0\}.$$

Then it is clear that M is ruled if and only if G_1 is empty.

Lemma 2 *Let M be a real hypersurface in $M_n(c)$. Then*

(a) $T\xi = \operatorname{grad}\alpha + 2AV - \alpha V$,
(b) $(\nabla_\xi A)\xi = T\xi + \alpha V = \operatorname{grad}\alpha + 2AV$.

Proof For any $X \in \Gamma(TM)$, the Codazzi equation (5) and (6) imply that

$$\begin{aligned}\langle TX, \xi\rangle &= \langle(\nabla_X A)\xi, \xi\rangle - \alpha\langle X, V\rangle \\ &= \langle\nabla_X A\xi, \xi\rangle - \langle A\nabla_X\xi, \xi\rangle - \alpha\langle X, V\rangle \\ &= X\alpha - 2\langle\nabla_X\xi, A\xi\rangle - \alpha\langle X, V\rangle \\ &= X\alpha + 2\langle AX, V\rangle - \alpha\langle X, V\rangle.\end{aligned}$$

Hence we have proved Statement (a). Statement (b) can be obtained directly from (6), the Codazzi equation and Statement (a). □

Lemma 3 *Let M be a ruled hypersurface in $M_n(c)$. Then*

$$\langle TX, Y\rangle = 0 \tag{7}$$

for all $X, Y \in \Gamma(\mathscr{D})$.

Proof By Lemma 1, we have $\langle AX, Y\rangle = 0$, for any $X, Y \in \Gamma(\mathscr{D})$. By differentiating covariantly both sides of this equation in the direction of ξ, we obtain

$$\begin{aligned}0 &= \langle(\nabla_\xi A)X, Y\rangle + \langle A\nabla_\xi X, Y\rangle + \langle AX, \nabla_\xi Y\rangle \\ &= \langle(\nabla_\xi A)X, Y\rangle + \eta(\nabla_\xi X)\eta(AY) + \eta(AX)\eta(\nabla_\xi Y).\end{aligned}$$

It follows from (5) and the definitions of V and T that we obtain (7). □

3 A characterization of certain ruled real hypersurfaces in $M_n(c)$

Throughout this section, suppose M is a connected real hypersurface in $M_n(c)$, $n \geq 3$. We also suppose that M satisfies the condition (2), i.e.,

$$\langle(\phi A - A\phi)X, Y\rangle = 0$$

for any $X, Y \in \Gamma(\mathscr{D})$. We remark that the condition (2) is equivalent to

$$(\phi A - A\phi)X = \eta(X)V + \langle X, V\rangle\xi \tag{8}$$

for any $X \in \Gamma(TM)$.

Lemma 4 *For any $X \in \Gamma(TM)$,*

$$\nabla_X V = -A^2X + \alpha AX + \langle X, V\rangle V - c\phi^2 X + \phi TX.$$

Proof For any $X \in \Gamma(TM)$, by using the Codazzi equation and (5), we get

$$\begin{aligned}\nabla_X V =&(\nabla_X \phi)A\xi + \phi(\nabla_X A)\xi + \phi A \nabla_X \xi\\ =&\alpha AX - \eta(A^2X)\xi + \eta(AX)\phi V + \langle X, V\rangle V - c\phi^2 X + \phi TX + \phi A \phi AX.\end{aligned}$$

Next, by using (4) and (8), we obtain the Lemma. □

Lemma 5 *For any $X \in \Gamma(TM)$,*

$$(\phi T - T\phi)X = \langle \phi T\xi, X\rangle \xi + \eta(X)\phi T\xi. \tag{9}$$

Proof For any $X \in \Gamma(TM)$, by differentiating covariantly both sides of the equation (8) in the direction of ξ, we obtain

$$\begin{aligned}&(\nabla_\xi \phi)AX + \phi(\nabla_\xi A)X - (\nabla_\xi A)\phi X - A(\nabla_\xi \phi)X\\ &= \langle X, \nabla_\xi \xi\rangle V + \eta(X)\nabla_\xi V + \langle \nabla_\xi V, X\rangle \xi + \langle X, V\rangle \nabla_\xi \xi.\end{aligned}$$

By using (5), (6) and Lemma 4, this equation becomes

$$\begin{aligned}&2\eta(AX)A\xi + \eta(AX)\phi^2 A\xi + \phi TX - \langle \phi X, V\rangle A\xi - T\phi X\\ &= \alpha\eta(X)A\xi + \eta(X)\phi T\xi + \alpha\eta(AX)\xi + \langle \phi T\xi, X\rangle \xi.\end{aligned}$$

Next, by using (4) and this equation, we obtain (9). □

Lemma 6 *For any $Y, Z \in \Gamma(\mathscr{D})$,*

$$(\nabla_Y A)Z = \langle AY, Z\rangle V + \langle Y, V\rangle AZ + \langle Z, V\rangle AY + \{\langle TY, Z\rangle - c\langle \phi Y, Z\rangle\}\xi. \tag{10}$$

Proof For any $Y, Z \in \Gamma(\mathscr{D})$, from [13, Lemma 2.1], we have

$$-\phi^2(\nabla_Y A)Z = \langle AY, Z\rangle V - \langle Y, V\rangle \phi^2 AZ - \langle Z, V\rangle \phi^2 AY.$$

By taking this into account,

$$(\nabla_Y A)Z = -\phi^2(\nabla_Y A)Z + \eta\left((\nabla_Y A)Z\right)\xi$$

and then using the Codazzi equation and (6), we obtain equation (10). □

Lemma 7 *For any $X, Y, Z \in \Gamma(\mathscr{D})$,*

(a) $\langle TY, Z\rangle = 0$,
(b) $\langle (\nabla_X T)Y, Z\rangle = \langle (\nabla_Y T)X, Z\rangle$.

Proof For any $Y,Z \in \Gamma(\mathscr{D})$, by differentiating both sides of (10) covariantly in the direction of $X \in \Gamma(\mathscr{D})$, we obtain

$$\begin{aligned}
&(\nabla^2_{XY}A)Z + (\nabla_{\nabla_X Y}A)Z + (\nabla_Y A)\nabla_X Z \\
&\quad = \langle(\nabla_X A)Y,Z\rangle V + \langle A\nabla_X Y,Z\rangle V + \langle AY,\nabla_X Z\rangle V + \langle AY,Z\rangle \nabla_X V \\
&\qquad + \langle Y,V\rangle(\nabla_X A)Z + \langle Y,V\rangle A\nabla_X Z + \langle \nabla_X Y,V\rangle AZ + \langle Y,\nabla_X V\rangle AZ \\
&\qquad + \langle Z,V\rangle(\nabla_X A)Y + \langle Z,V\rangle A\nabla_X Y + \langle \nabla_X Z,V\rangle AY + \langle Z,\nabla_X V\rangle AY \\
&\qquad + \langle(\nabla_X T)Y,Z\rangle\xi + \langle T\nabla_X Y,Z\rangle\xi + \langle TY,\nabla_X Z\rangle\xi + \langle TY,Z\rangle\nabla_X\xi \\
&\qquad - c\langle(\nabla_X\phi)Y,Z\rangle\xi - c\langle\phi\nabla_X Y,Z\rangle\xi - c\langle\phi Y,\nabla_X Z\rangle\xi - c\langle\phi Y,Z\rangle\nabla_X\xi.
\end{aligned}$$

By using (5), (6) and (10), this equation yields

$$\begin{aligned}
(\nabla^2_{XY}A)Z = {} & \langle(\nabla_X A)Y,Z\rangle V + \langle Y,V\rangle(\nabla_X A)Z + \langle Z,V\rangle(\nabla_X A)Y + \langle(\nabla_X T)Y,Z\rangle\xi \\
& - \langle Y,\phi AX\rangle\phi^2 TZ - \langle Z,\phi AX\rangle\phi^2 TY - c\langle Z,\phi AX\rangle\phi Y - c\langle\phi Y,Z\rangle\phi AX \\
& + \langle AY,Z\rangle\nabla_X V + \langle Y,\nabla_X V\rangle AZ + \langle Z,\nabla_X V\rangle AY + \langle TY,Z\rangle\phi AX.
\end{aligned}$$

It follows from (2), Lemma 5, the Codazzi equation and (10) that

$$\begin{aligned}
&(R(X,Y)A)Z \\
&= (\nabla^2_{XY}A)Z - (\nabla^2_{YX}A)Z \\
&= \{-c\langle Y,V\rangle\langle\phi X,Z\rangle + \langle Y,V\rangle\langle TX,Z\rangle + \langle(\nabla_X T)Y,Z\rangle \\
&\quad + c\langle X,V\rangle\langle\phi Y,Z\rangle - \langle X,V\rangle\langle TY,Z\rangle - \langle(\nabla_Y T)X,Z\rangle - 2c\langle Z,V\rangle\langle\phi X,Y\rangle\}\xi \\
&\quad - 2\langle Y,\phi AX\rangle\phi^2 TZ - \langle Z,\phi AX\rangle\phi^2 TY + \langle Z,\phi AY\rangle\phi^2 TX \\
&\quad + c\{-\langle Z,\phi AX\rangle\phi Y + \langle Z,\phi AY\rangle\phi X\} \\
&\quad - \langle AY,Z\rangle A^2 X + c\langle AY,Z\rangle X + \langle AY,Z\rangle\phi TX \\
&\quad + \langle AX,Z\rangle A^2 Y - c\langle AX,Z\rangle Y - \langle AX,Z\rangle\phi TY + 2\langle\phi TX,Y\rangle AZ \\
&\quad - \langle AX,AZ\rangle AY + c\langle X,Z\rangle AY + \langle\phi TX,Z\rangle AY \\
&\quad + \langle AY,AZ\rangle AX - c\langle Y,Z\rangle AX - \langle\phi TY,Z\rangle AX \\
&\quad - c\langle\phi Y,Z\rangle\phi AX + \langle TY,Z\rangle\phi AX + c\langle\phi X,Z\rangle\phi AY - \langle TX,Z\rangle\phi AY.
\end{aligned}$$

By applying the Gauss equation and (2) in the above equation, we get

$$\begin{aligned}
&\{\langle(\nabla_X T)Y,Z\rangle - \langle(\nabla_Y T)X,Z\rangle\}\xi - 2\langle Y,\phi AX\rangle\phi^2 TZ - \langle Z,\phi AX\rangle\phi^2 TY \\
&+ \langle Z,\phi AY\rangle\phi^2 TX + \langle TY,Z\rangle A\phi X - \langle TX,Z\rangle A\phi Y + 2\langle Y,\phi TX\rangle AZ \\
&+ \langle Z,\phi TX\rangle AY - \langle Z,\phi TY\rangle AX + \langle AY,Z\rangle\phi TX - \langle AX,Z\rangle\phi TY = 0. \qquad (11)
\end{aligned}$$

By taking inner product of each side of this equation with $W \in \Gamma(\mathscr{D})$, we obtain

$$\begin{aligned}
&2\langle Y,\phi AX\rangle\langle TZ,W\rangle + \langle Z,\phi AX\rangle\langle TY,W\rangle - \langle Z,\phi AY\rangle\langle TX,W\rangle + \langle W,\phi AX\rangle\langle TY,Z\rangle \\
&- \langle W,\phi AY\rangle\langle TX,Z\rangle + 2\langle Y,\phi TX\rangle\langle AZ,W\rangle + \langle Z,\phi TX\rangle\langle AY,W\rangle \\
&- \langle Z,\phi TY\rangle\langle AX,W\rangle + \langle W,\phi TX\rangle\langle AY,Z\rangle - \langle W,\phi TY\rangle\langle AX,Z\rangle = 0.
\end{aligned}$$

Now, by replacing Y,Z and W cyclically in the above equation and then summing the three obtained equations, with the help of (2) and Lemma 5, we obtain

$$\begin{aligned}
&\langle Y,\phi AX\rangle\langle TZ,W\rangle + \langle Z,\phi AX\rangle\langle TW,Y\rangle + \langle W,\phi AX\rangle\langle TY,Z\rangle \\
&+ \langle Y,\phi TX\rangle\langle AZ,W\rangle + \langle Z,\phi TX\rangle\langle AW,Y\rangle + \langle W,\phi TX\rangle\langle AY,Z\rangle = 0.
\end{aligned}$$

By switching X to ϕX in the above equation, we obtain

$$\begin{aligned}&\langle Y,AX\rangle\langle TZ,W\rangle+\langle Z,AX\rangle\langle TW,Y\rangle+\langle W,AX\rangle\langle TY,Z\rangle\\&+\langle Y,TX\rangle\langle AZ,W\rangle+\langle Z,TX\rangle\langle AW,Y\rangle+\langle W,TX\rangle\langle AY,Z\rangle=0.\end{aligned}\tag{12}$$

Consider a point $x \in G_1$, there is a vector $X_0 \in \mathscr{D}_x$ such that $\langle AX_0, X_0\rangle \neq 0$. We first let $X = Y = Z = W = X_0$ in (12) to get $\langle AX_0, X_0\rangle\langle TX_0, X_0\rangle = 0$, showing $\langle TX_0, X_0\rangle = 0$. Next, if we put $X = Y = Z = X_0$ in (12), then $\langle AX_0, X_0\rangle\langle TX_0, W\rangle = 0$ for any $W \in \mathscr{D}_x$ and so $\phi TX_0 = 0$. Finally, by putting $X = Y = X_0$ in (12), we have $\langle TZ, W\rangle = 0$ for any $Z, W \in \mathscr{D}_x$. Hence, $\phi T\phi = 0$ at such a point x.

On the other hand, the open submanifold $\mathrm{Int}(M - G_1)$ is a ruled hypersurface in $M_n(c)$ by virtue of Lemma 1 and so it follows from Lemma 3 that $\phi T\phi = 0$ on the open submanifold $\mathrm{Int}(M - G_1)$. Therefore, from the continuity of $||\phi T\phi||$, we conclude that $\phi T\phi = 0$ on the whole of M.

Statement (b) can be obtained directly from (11) and Statement (a). □

Lemma 8 *For any* $X, Y \in \Gamma(TM)$,

$$\begin{aligned}(\nabla_X A)Y=\langle AX,Y\rangle V+\langle X,V\rangle AY+\langle Y,V\rangle AX-c\{\eta(Y)\phi X+\langle\phi X,Y\rangle\xi\}\\+\{\eta(X)\eta(TY)+\eta(Y)\eta(TX)\}\xi+\eta(X)\eta(Y)\{T\xi-2(\xi\alpha)\xi\}.\end{aligned}\tag{13}$$

Proof First, from Lemma 6 and Lemma 7, we have

$$(\nabla_X A)Y=\langle AX,Y\rangle V+\langle X,V\rangle AY+\langle Y,V\rangle AX-c\langle\phi X,Y\rangle\xi$$

for any $X, Y \in \Gamma(\mathscr{D})$. Then by using this equation, (6), Lemma 2 and Lemma 7, equation (13) can be obtained after verifying each of the $\mathscr{D}$- or $\mathrm{Span}\{\xi\}$-components for both X and Y in (13). □

Lemma 9 *(a)* $||\phi A\phi||\,\phi T\xi = 0$,
(b) $\langle(\nabla_X T)Y, Z\rangle = 0, \quad \forall X, Y, Z \in \Gamma(\mathscr{D})$.

Proof For any $Y, Z \in \Gamma(\mathscr{D})$, by differentiating both sides of Lemma 7(a) in the direction of $X \in \Gamma(\mathscr{D})$ and then using (5), we get

$$\langle(\nabla_X T)Y,Z\rangle-\langle Y,\phi AX\rangle\eta(TZ)-\langle Z,\phi AX\rangle\eta(TY)=0.\tag{14}$$

This, together with (2) and Lemma 7(b), give

$$2\langle Y,\phi AX\rangle\eta(TZ)+\langle Z,\phi AX\rangle\eta(TY)-\langle Z,\phi AY\rangle\eta(TX)=0.\tag{15}$$

Statement (a) clearly holds for a point $x \in M - G_1$. Hence, we consider a point $x \in G_1$ and a unit vector $X_0 \in \mathscr{D}_x$ such that $\langle AX_0, X_0\rangle \neq 0$. First, by putting $Y = \phi X_0$ and $Z = X = X_0$ in (15), we get $\eta(TX_0) = 0$. From this and after we put $Y = Z = \phi X_0$ and $X = X_0$ in (15), then $\eta(T\phi X_0) = 0$. Finally, if we put $Z = \phi X_0$ and $X = X_0$ in (15), we have $\eta(TY) = 0$, for any $Y \in \mathscr{D}_x$ and so $\phi T\xi = 0$. This proves Statement (a).

Statement (a) implies that the second and third terms in (14) vanish everywhere. This proves Statement (b). □

Lemma 10 $\xi\alpha = \eta(T\xi) = 0$ *and* $\mathrm{grad}\,\alpha = \alpha V - 2AV$ *on* G_1.

Proof Since $||\phi A\phi|| \neq 0$ on G_1, Lemma 2(a) and Lemma 9(a) imply that

$$Y\alpha = \alpha\langle Y,V\rangle - 2\langle AY,V\rangle, \tag{16}$$

for all $Y \in \Gamma(\mathscr{D})$ and so

$$\begin{aligned} XY\alpha =&(X\alpha)\langle Y,V\rangle + \alpha\langle \nabla_X Y,V\rangle + \alpha\langle Y,\nabla_X V\rangle \\ &-2\langle(\nabla_X A)Y,V\rangle - 2\langle A\nabla_X Y,V\rangle - 2\langle AY,\nabla_X V\rangle \end{aligned}$$

for any $X,Y \in \Gamma(\mathscr{D})$. By making use of Lemma 4, Lemma 7, the Codazzi and the above equations, we have

$$[X,Y]\alpha = \alpha\langle [X,Y],V\rangle - 2\langle A[X,Y],V\rangle.$$

Therefore, by virtue of (4) and (16), we obtain

$$\langle [X,Y],\xi\rangle\xi\alpha = \langle [X,Y],\xi\rangle\eta(T\xi) = 0.$$

Since $\mathscr{D}$ is not integrable on G_1, $\langle [X,Y],\xi\rangle \neq 0$ for some $X,Y \in \Gamma(\mathscr{D})$. This proves $\xi\alpha = 0$ and furthermore, together with (16), yield $\operatorname{grad}\alpha = \alpha V - 2AV$. □

Lemma 11 *Let h $(:= \mathrm{Trace}\,A)$ be the mean curvature of M. Then* $\operatorname{grad}(h-\alpha) = (h-\alpha)V + 4AV$. *Furthermore, if M is minimal then $AV = 0$ and* $\operatorname{grad}\alpha = \alpha V$.

Proof Let $E_1, E_2 \cdots, E_{2n-1}$ be a local field of orthonormal vectors in $\Gamma(TM)$, and $X \in \Gamma(TM)$. Then with the help of Lemma 2(a) and Lemma 8, we have

$$Xh = \sum_{j=1}^{2n-1} \langle(\nabla_X A)E_j,E_j\rangle = \langle 4AV + hV - \alpha V,X\rangle + X\alpha.$$

Hence, $\operatorname{grad}(h-\alpha) = (h-\alpha)V + 4AV$.

Now suppose M is minimal, i.e., $h = 0$. Then

$$\operatorname{grad}\alpha = \alpha V - 4AV.$$

For $x \in G_1$, by Lemma 10 and the above equation, we obtain $AV = 0$ and so $\operatorname{grad}\alpha = \alpha V$ on the closure of G_1. On the other hand, it is clear that $\alpha = 0$ and $AV = 0$ on $\mathrm{Int}(M - G_1)$ as it is an open part of a ruled real hypersurface. This completes the proof. □

Now we are in a position to prove Theorem 2.

Proof (of Theorem 2) If $d\alpha(\xi)$ is nowhere zero in an open dense subset of M then by the Lemma 10, G_1 is nowhere dense. Since G_1 is open, it must be empty hence M is ruled according to Lemma 1. □

Remark 1 Theorem 2 is not true if the condition "$d\alpha(\xi)$ is nowhere zero in an open dense subset of M" is removed. In the next section, we will see that there exists certain non-ruled real hypersurfaces satisfying (2) but with $d\alpha(\xi) = 0$ everywhere. However, there do exist ruled real hypersurfaces that satisfy the hypotheses in Theorem 2. For instance, the ruled real hypersurfaces given in [7]. We will briefly discuss its construction, and one may refer to [7] for detail.

Recall that a Legendre curve $\gamma(t) = (\Gamma_1(t),\Gamma_2(t))$ in $S^3 \subset \mathbb{C}^2$, $t \in I$, satisfies the differential equation

$$\gamma''(t) - i\omega(t)\gamma'(t) + \gamma(t) = 0$$

for some nonzero real-valued function ω (cf. [6]).

Let $\gamma(t) = (\Gamma_1(t), \Gamma_2(t))$ in S^3, $t \in I$, be a unit speed Legendre curve. Then the mapping

$$\Psi(z_1, \cdots, z_{n-1}, t) = (\Gamma_1(t)z_1, \Gamma_2(t)z_1, z_2, \cdots, z_{n-1}), \quad z_1 \neq 0$$

defines a ruled real hypersurface in $\mathbb{C}^n$. The squared mean curvature of the real hypersurface is given by

$$h^2 = \frac{\omega^2}{|z_1|^2} \ (= \alpha^2).$$

If the function ω is selected in such a way that $\omega'(t) \neq 0$ for all t then from the above equation, we may verify that $\xi\alpha \neq 0$ for such a ruled real hypersurface.

By using a similar argument in the real hypersurfaces constructed in [7] for the complex projective and hyperbolic ambient spaces, one may obtain ruled real hypersurfaces with $\xi\alpha \neq 0$.

4 Auxiliary lemmas

In this section, suppose that M is a connected real hypersurface in $M_n(c)$, $n \geq 3$, satisfying (2), i.e.,

$$\langle(\phi A - A\phi)X, Y\rangle = 0$$

for any $X, Y \in \Gamma(\mathscr{D})$. We further suppose that $\beta \neq 0$ everywhere and

$$\operatorname{grad}\alpha = \alpha V. \tag{17}$$

We denote by $U := -\beta^{-1}\phi V$, then we have

$$A\xi = \alpha\xi + \beta U. \tag{18}$$

Under these assumptions, we can obtain the following lemma, giving us some information about the principal curvatures of M.

Lemma 12 *(a)* $A\phi U = 0$ *and* $AU = \beta\xi$,
(b) $\nabla_X V = -A^2X + \alpha AX + \langle X, V\rangle V - c\phi^2 X$,
(c) $(\nabla_X A)Y = \langle AX, Y\rangle V + \langle X, V\rangle AY + \langle Y, V\rangle AX - c\{\eta(Y)\phi X + \langle\phi X, Y\rangle\xi\}$,
for any $X, Y \in \Gamma(TM)$.

Proof First, for $x \in G_1$, by (17) and Lemma 10, we see that $A\phi U = 0$ at the point x. Next, since $\mathrm{Int}(M - G_1)$ is an open part of a ruled real hypersurface, $A\phi U = 0$ on $\mathrm{Int}(M - G_1)$ and hence, we conclude that $A\phi U = 0$ everywhere. Finally, by taking into account that $A\phi U = 0$ and (2), we have $AU = \beta\xi$.

Next, it follows from Lemma 2, (17) and Statement (a) that $T\xi = 0$. This, together with Lemma 7, gives $T = 0$. Statements (b) and (c), hence can be deduced from Lemma 4 and Lemma 8 respectively. □

By verifying the ϕU- and $(\phi U)^\perp$-components in Lemma 12(b), we obtain

Lemma 13 *(a)* $\operatorname{grad}\beta = (c + \beta^2)\phi U$
(b) $\beta\nabla_X\phi U = -A^2X + \alpha AX - c\{\phi^2 X + \langle X, \phi U\rangle\phi U\}$.

Since $\mathrm{Span}\{\xi, U, \phi U\}$ is invariant by A and ϕ, together with (2), we have

Lemma 14 *For any $X \perp \xi, U$ and ϕU, $AX = \lambda X \iff A\phi X = \lambda \phi X$.*

Lemma 15 *For any $X \in \Gamma(TM)$,*

$$A^3X - \alpha A^2X - (c+\beta^2)AX + cA(\eta(X)\xi + \langle X, U\rangle U) = 0.$$

Proof By putting $Y = V$ in Lemma 12(c) and taking into account that $AV = 0$, we have $-A\nabla_X V = \langle V, V\rangle AX - c\langle \phi X, V\rangle \xi$. Then applying Lemma 12(a) and (b), we obtain the equation. □

By taking into account the fact that $\mathrm{Span}\{\xi, U\}$ is A-invariant in Lemma 15, we have the following lemma.

Lemma 16 *Let X be a unit vector tangent to M. If $AX = \lambda X$ then*

(a) $X \in \mathrm{Span}\{\xi, U\}^\perp \Rightarrow \lambda(\lambda^2 - \alpha\lambda - (c+\beta^2)) = 0$,
(b) $X \in \mathrm{Span}\{\xi, U\} \Rightarrow \lambda^2 - \alpha\lambda - \beta^2 = 0$.

5 An example of a real hypersurface in $\mathbb{C}^n$

The objective of this section is to give an example of a real hypersurface in $\mathbb{C}^n$ satisfying (2).

Let $\hat{a}, \hat{b} > 0$ with $\hat{a}^2 + \hat{b}^2 = 1$ and p_0, p_1, p_2 non-negative integers with $p_0 + p_1 + p_2 + 2 = n$. Consider

$$\Psi : \mathbb{C}^{p_0} \times \mathbb{R}^+ \times S^{2p_1+1}(\hat{a}) \times S^{2p_2+1}(\hat{b}) \to \mathbb{C}^n = \mathbb{C}^{p_1+1} \times \mathbb{C}^{p_2+1} \times \mathbb{C}^{p_0}$$

given by

$$\Psi(z, r, x_1, x_2) = (rx_1, rx_2, z). \tag{19}$$

Then the mapping Ψ defines a real hypersurface M in $\mathbb{C}^n$.

Let $(u_1, \cdots, u_{2p_1+1})$ and $(v_1, \cdots, v_{2p_2+1})$ be local coordinates for $S^{2p_1+1}(\hat{a})$ and $S^{2p_2+1}(\hat{b})$ respectively. Then the tangent space T_wM at the point $w = \Psi(z, r, x_1, x_2)$ is spanned by the vectors

$$\begin{aligned}
&\partial_r \Psi = (x_1, x_2, 0);\\
&\partial_{u_j} \Psi = (r\partial_{u_j} x_1, 0, 0); \qquad j \in \{1, \cdots, 2p_1+1\}\\
&\partial_{v_\tau} \Psi = (0, r\partial_{v_\tau} x_2, 0); \qquad \tau \in \{1, \cdots, 2p_2+1\}\\
&\partial_{z_a} \Psi = (0, 0, e_a), \quad i\partial_{z_a} \Psi = (0, 0, ie_a); \qquad a \in \{1, \cdots, p_0\}.
\end{aligned}$$

where $e_1 = (1, 0, \cdots, 0) \in \mathbb{C}^{p_0}$, *etc*. From the above equations, we may regard $M = \mathbb{C}^{p_0} \times \mathbb{R}^+ \times_r (S^{2p_1+1}(\hat{a}) \times S^{2p_2+1}(\hat{b}))$. We let $U = i\partial_r\Psi = (ix_1, ix_2, 0)$ and define a unit vector normal to M by

$$N = \left(-\frac{\hat{b}}{\hat{a}}x_1, \frac{\hat{a}}{\hat{b}}x_2, 0\right).$$

Hence, the structure vector is given by

$$\xi = -iN = \left(\frac{\hat{b}}{\hat{a}} ix_1, -\frac{\hat{a}}{\hat{b}} ix_2, 0\right).$$

Further, we consider the following subspaces

$$\begin{aligned}
\Lambda_0 &= \mathrm{Span}\{\phi U\} \oplus \hat{\Lambda}_0, \quad \hat{\Lambda}_0 = \mathrm{Span}\{\partial_{z_a}\Psi, i\partial_{z_a}\Psi : 1 \le a \le p_0\},\\
\Lambda_1 &= \mathrm{Span}\{\partial_{u_j}\Psi : 1 \le j \le 2p_1+1\}, \quad \hat{\Lambda}_1 = \{X \in \Lambda_1 : X \perp \xi, U\},\\
\Lambda_2 &= \mathrm{Span}\{\partial_{v_\tau}\Psi : 1 \le \tau \le 2p_2+1\}, \quad \hat{\Lambda}_2 = \{X \in \Lambda_2 : X \perp \xi, U\}.
\end{aligned}$$

By a direct computation, we can see that

$$\begin{aligned}
AX &= \lambda_j X, \quad \forall X \in \Lambda_j;\\
A\xi &= \alpha\xi + \beta U, \quad AU = \beta\xi, \quad A\phi U = 0,
\end{aligned} \tag{20}$$

where $\lambda_0 = 0$, $\lambda_1 = \hat{b}/\hat{a}r$, $\lambda_2 = -\hat{a}/\hat{b}r$, $\alpha = (\hat{b}/\hat{a} - \hat{a}/\hat{b})/r$ and $\beta = 1/r$. By looking at (20) and taking into account that the subspaces $\hat{\Lambda}_0$, $\hat{\Lambda}_1$ and $\hat{\Lambda}_2$ are invariant by ϕ, we have

$$(\phi A - A\phi)X = 0,$$

for any $X \in \hat{\Lambda}_0 \oplus \hat{\Lambda}_1 \oplus \hat{\Lambda}_2$. Further, we can also see that

$$(\phi A - A\phi)U = 0, \quad (\phi A - A\phi)\phi U = \beta\xi.$$

All these amount to say that the condition (2) is satisfied, i.e., we have

$$\langle \phi AX - A\phi X, Y\rangle = 0,$$

for all $X \in \mathscr{D}_w$ at the point $w = \Psi(z, r, x_1, x_2)$.

Since α is a real-valued function depending only on r and $d\alpha/dr = -\alpha/r$, we have $\operatorname{grad}\alpha = \alpha\beta\phi U$ and so $\xi\alpha = 0$. Furthermore, we can see that M is ruled if and only if $\hat{\Lambda}_1 = \hat{\Lambda}_2 = 0$, or equivalently, $p_1 = p_2 = 0$.

M has three principal curvatures λ_j, as described above, with multiplicities $2p_j + 1$, $j \in \{0,1,2\}$. Hence M is minimal implies that $(2p_1+1)(\hat{b}/\hat{a}) - (2p_2+1)(\hat{a}/\hat{b}) = 0$ and so

$$\hat{a} = \sqrt{\frac{2p_1+1}{2p_1+2p_2+2}}, \quad \hat{b} = \sqrt{\frac{2p_2+1}{2p_1+2p_2+2}}.$$

We summarize all these observations in the following theorem.

Theorem 7 *Let M be a real hypersurface in $\mathbb{C}^n$, determined by the immersion (19). Then*

(a) M satisfies (2);
(b) $\operatorname{grad}\alpha = \alpha V$;
(c) M is ruled if and only if $p_1 = p_2 = 0$;
(d) M is minimal if and only if

$$\hat{a} = \sqrt{\frac{2p_1+1}{2p_1+2p_2+2}}, \quad \hat{b} = \sqrt{\frac{2p_2+1}{2p_1+2p_2+2}}.$$

6 A class of real hypersurfaces in $\mathbb{C}^n$

In this section we consider connected real hypersurfaces M in the complex Euclidean space $\mathbb{C}^n$, satisfying the hypotheses in Sect. 4, i.e., for the case $c = 0$, $\beta \neq 0$ everywhere, $\operatorname{grad}\alpha = \alpha V$ and

$$\langle(\phi A - A\phi)X, Y\rangle = 0$$

for any $X, Y \in \Gamma(\mathscr{D})$.

Lemma 17 *The principal curvatures λ of M are characterized by the equation $\lambda(\lambda^2 - \alpha\lambda - \beta^2) = 0$. Furthermore, there are exactly three distinct principal curvatures at each point of M.*

Proof By virtue of Lemma 16, we see that the principal curvatures satisfy the stated equation. Furthermore, it follows from this equation that there are at most three distinct principal curvatures at each point. Since $A\phi U = 0$, $\lambda_0 = 0$ is one of the principal curvatures. Since $\operatorname{Span}\{\xi, U\}$ is A-invariant, there are orthonormal vectors $Y_j \in \operatorname{Span}\{\xi, U\}$ such that $AY_j = \lambda_j Y_j$, where $\lambda_j^2 - \alpha\lambda_j - \beta^2 = 0$, for $j \in \{1,2\}$. If $\lambda_1 = \lambda_2$ then ξ is principal and so $\beta = 0$. This is a contradiction and the proof is completed. □

Let $\lambda_0 = 0$, λ_1 and λ_2 be the principal curvatures of M. Then it follows from Lemma 17 that the principal curvatures are differentiable on M and with constant multiplicities.

Denote by Λ_j the subbundle of TM foliated by A corresponding to λ_j, for $j \in \{0,1,2\}$. It follows from (18) and Lemma 12 that $\operatorname{Span}\{\xi, U, \phi U\}$ is invariant by A, $\phi U \in \Lambda_0$, $\beta U + \lambda_1\xi \in \Lambda_1$ and $\beta U + \lambda_2\xi \in \Lambda_2$. Together with Lemma 14, we see that Λ_j is of odd dimension. Hence, we have

Lemma 18 $\dim\Lambda_j = 2p_j + 1$, *for $j \in \{0,1,2\}$, where p_0, p_1 and p_2 are non-negative integers with $p_0 + p_1 + p_2 + 2 = n$.*

Lemma 19 $\operatorname{grad}\lambda_j = \lambda_j\beta\phi U$, *for $j \in \{0,1,2\}$.*

Proof It is trivial for $j = 0$. For $j \neq 0$, we consider a unit vector field $Y \in \Gamma(\Lambda_j)$. It follows from Lemma 12 that

$$X\lambda_j = \langle(\nabla_X A)Y, Y\rangle = \lambda_j\langle X, V\rangle = \lambda_j\beta\langle X, \phi U\rangle,$$

for any $X \in \Gamma(TM)$. This completes the proof. □

Lemma 20

$$\nabla_X\phi U = \begin{cases} 0, & X \in \Gamma(\Lambda_0) \\ -\beta X, & X \in \Gamma(\Lambda_1 \oplus \Lambda_2) \end{cases}$$

Proof It follows from Lemma 13 that $\nabla_X\phi U = 0$, for any $X \in \Gamma(\Lambda_0)$. Next, for $X \in \Gamma(\Lambda_1 \oplus \Lambda_2)$, then as $\lambda_1, \lambda_2 \neq 0$, Lemma 16 imply that $-A^2X + \alpha AX = -\beta^2 X$ and then by virtue of Lemma 13 again, we obtain the lemma. □

From Lemma 12 and Lemma 20, we see that $\bar{\nabla}_{\phi U}\phi U = 0$, where $\bar{\nabla}$ is the Levi-Civita connection on $\mathbb{C}^n$, hence each of the integral curves of ϕU is a geodesic in $\mathbb{C}^n$. Hence, we have

Lemma 21 *Each integral curve of ϕU is a straight line in $\mathbb{C}^n$.*

We first consider, at the moment, the simplest case, $\dim \Lambda_0 = 1$, i.e., $\Lambda_0 = \mathrm{Span}\{\phi U\}$, and then we shall extend the result to the general case $\dim \Lambda_0 > 1$.

Lemma 22 *Suppose* $\dim \Lambda_0 = 1$. *Then the distributions* Λ_1, Λ_2 *and* $\Lambda_1 \oplus \Lambda_2$ *are integrable, and each leaf* M_+ *of* $\Lambda_1 \oplus \Lambda_2$ *is immersed as a totally umbilical hypersurface in* M. *Furthermore, if* M_1 *and* M_2 *are respectively leaves for* Λ_1 *and* Λ_2 *then* M_+ *is locally a Riemannian product* $M_1 \times M_2$.

Proof For any $X_1, Y_1 \in \Gamma(\Lambda_1)$ and $X_2 \in \Gamma(\Lambda_2)$, by Lemma 12 and Lemma 19, we have

$$\lambda_1 \nabla_{X_1} Y_1 - A \nabla_{X_1} Y_1 = \lambda_1 \langle X_1, Y_1 \rangle V \tag{21}$$

$$\lambda_2 \nabla_{X_1} X_2 - A \nabla_{X_1} X_2 = 0. \tag{22}$$

The equation (21) implies that $A[X_1, Y_1] = \lambda_1 [X_1, Y_1]$ and so $[X_1, Y_1] \in \Gamma(\Lambda_1)$, or Λ_1 is integrable. Similarly, we can show that Λ_2 is integrable.

On the other hand, for any $X, Y \in \Gamma(\Lambda_1 \oplus \Lambda_2)$, it follows from Lemma 20 that

$$\langle \nabla_X Y, \phi U \rangle = -\langle \nabla_X \phi U, Y \rangle = \beta \langle X, Y \rangle.$$

This imply that $[X, Y] \perp \phi U$ and hence $\Lambda_1 \oplus \Lambda_2$ is integrable. Let ∇^+ be the Levi-Civita connection on M_+ induced by ∇. Then

$$\nabla_X Y = \nabla^+_X Y + \langle \nabla_X Y, \phi U \rangle \phi U = \nabla^+_X Y + \beta \langle X, Y \rangle \phi U. \tag{23}$$

Therefore, M_+ is a totally umbilical hypersurface in M.

Next, it follows from (22) that $\nabla_{X_1} X_2 \in \Gamma(\Lambda_2)$ and so $\nabla_{X_1} Y_1 \perp \Gamma(\Lambda_2)$. Let ∇^1 denotes the connection induced on M_1. Then

$$\nabla_{X_1} Y_1 = \nabla^1_{X_1} Y_1 + \beta \langle X_1, Y_1 \rangle \phi U. \tag{24}$$

By comparing (23) and (24), we have $\nabla^+_{X_1} Y_1 = \nabla^1_{X_1} Y_1$, showing that M_1 is totally geodesic in M_+. Similarly, we can also see that M_2 is totally geodesic in M_+. Accordingly, M_+ is locally a Riemannian product $M_1 \times M_2$. □

For each pair $(x_1, x_2) \in M_1 \times M_2$, by Lemma 21, the integral curve γ of $-\phi U$ is a straight line in $\mathbb{C}^n$. Hence after a translation, we may write

$$\gamma(r) = -r\phi U, \quad (r > 0).$$

On the other hand, let $\bar{\nabla}$ be the Levi-Civita connection on $\mathbb{C}^n$. Since $\bar{\nabla}_{\phi U} \phi U = 0$ (by Lemma 12 and Lemma 20), ϕU is independent of r. Therefore we have

Lemma 23 *Suppose* $\dim \Lambda_0 = 1$ *and* $\Psi(r, x_1, x_2)$ *is an isometric immersion of* M *into* $\mathbb{C}^n$. *Then we have*

$$\Psi(r, x_1, x_2) = -r\phi U(x_1, x_2), \quad (x_1 \in M_1, x_2 \in M_2).$$

Lemma 24 *Suppose* $\dim \Lambda_0 = 1$. *Then* $\beta = r^{-1}$, $\lambda_1 = ba^{-1}r^{-1}$, $\lambda_2 = -ab^{-1}r^{-1}$ *and* $\alpha = (ba^{-1} - ab^{-1})r^{-1}$, *where* a *and* b *are positive constants.*

Proof It follows from Lemma 13 that β is independent of x_1 and x_2, and $d\beta/dr = -\beta^2$. Therefore, by using an appropriate translation, we obtain $\beta = r^{-1}$. Similarly, by Lemma 19 and then after solving a first order differential equation, we see that $\lambda_j = \varepsilon_j r$ for some constant ε_j. On the other hand, Lemma 16 implies that $\lambda_1 \lambda_2 = -\beta^2$ and $\lambda_1 + \lambda_2 = \alpha$, hence we may put $\varepsilon_1 = ba^{-1}$ and $\varepsilon_2 = -ab^{-1}$ to obtain these expressions. □

Lemma 25 *Suppose* $\dim \Lambda_0 = 1$. *Then locally,* $M = \mathbb{R}^+ \times_f (M_1 \times M_2)$, *where* $f(r) = r$.

Proof Since M_0 $(= \mathbb{R}^+)$ is a geodesic in M and M^+ is totally umbilical in M, by a result of [11], M is locally a warped product $\mathbb{R}^+ \times_f (M_1 \times M_2)$ and the warping function f is determined by

$$-\operatorname{grad}\log f = \beta\phi U = \frac{1}{\beta}\operatorname{grad}\beta.$$

Therefore $f(r) = \omega r$, for some positive constant ω. By reparametrizing the coordinate of $\mathbb{R}^+$, we may obtain $\omega = 1$ so that $f(r) = r$. □

Theorem 8 *Let M be a real hypersurface in $\mathbb{C}^n$, $n \geq 3$. Suppose M satisfies (2) and* $\operatorname{grad}\alpha = \alpha V$. *If β is never vanishing in M and* $\dim \Lambda_0 = 1$ *then up to rigid motions of $\mathbb{C}^n$, M is defined by the immersion*

$$\Psi : \mathbb{R}^+ \times_r \left(S^{2p_1+1}(\hat{a}) \times S^{2p_2+1}(\hat{b})\right) \to \mathbb{C}^{p_1+1} \times \mathbb{C}^{p_2+1} = \mathbb{C}^n$$

given by

$$\Psi(r, x_1, x_2) = (rx_1, rx_2)$$

where $\hat{a}, \hat{b} > 0$ with $\hat{a}^2 + \hat{b}^2 = 1$ and $p_1 + p_2 + 2 = n$.

Proof Let r, $u = (u_1, \cdots, u_{2p_1+1})$ and $v = (v_1, \cdots, v_{2p_2+1})$ be local coordinates for M_0, M_1 and M_2 respectively. Suppose $w = \Psi(r, u, v)$ is an isometric immersion of M into $\mathbb{C}^n$. Define

$$\Psi^1 := \frac{-a(a\phi U + bN)}{a^2 + b^2}, \quad \Psi^2 := \frac{-b(b\phi U - aN)}{a^2 + b^2}.$$

Then $\Psi = r(\Psi^1 + \Psi^2)$ by Lemma 23. It follows from the Gauss and Weingarten formulas, Lemma 12, Lemma 20 and Lemma 24 that

$$\bar{\nabla}_X(a\phi U + bN) = a\bar{\nabla}_X \phi U + b\bar{\nabla}_X N = a\left(-\frac{1}{r}X\right) - b\left(-\frac{a}{b}\frac{1}{r}X\right) = 0$$

$$\bar{\nabla}_{\phi U}(a\phi U + bN) = 0$$

for any $X \in \Gamma(\Lambda_2)$. Therefore, Ψ_1 is independent of v and r. Similarly, we see that Ψ_2 is independent of u and r. Since $M = \mathbb{R}^+ \times_r (M_1 \times M_2)$, the mapping $x_1 = \Psi^1(u)$ (resp. $x_2 = \Psi^2(v)$) is an isometric immersion of M_1 (resp. M_2) into $\mathbb{C}^n$.

For any fixed pair r and v, without loss of generality, assume that $\Psi^2(v) = 0$ then we have $\Psi^1 = r^{-1}\Psi$. In the following, we write $\Psi_k = \partial\Psi/\partial u_k$, $\Psi_{kj} = \partial^2\Psi/\partial u_k \partial u_j$, *etc*, for $j, k \in \{1, \cdots, 2p_1 + 1\}$. Then we have $\Psi_k^1 = r^{-1}\Psi_k$ and

$$\bar{\nabla}_{\Psi_j^1}\Psi_k^1 = \Psi_{kj}^1 = \frac{1}{r}\Psi_{kj} = \frac{1}{r}\bar{\nabla}_{\Psi_j}\Psi_k.$$

Since $\Psi_j, \Psi_k \in \Gamma(\Lambda_1)$, by using the Gauss formula, Lemma 20 and Lemma 24, we have

$$\begin{aligned}\bar{\nabla}_{\Psi_j^1}\Psi_k^1 &= \nabla^1_{\Psi_j^1}\Psi_k^1 + \langle \Psi_j, \Psi_k \rangle \frac{1}{r^2 a}\{a\phi U + bN\} \\ &= \nabla^1_{\Psi_j^1}\Psi_k^1 - \langle \Psi_j^1, \Psi_k^1 \rangle \frac{1}{\hat{a}^2}\Psi^1\end{aligned}$$

where $\hat{a} = a(a^2 + b^2)^{-1/2}$ and $\nabla^1_{\Psi_j^1}\Psi_k^1$ denotes the Λ_1-component of $r^{-1}\nabla_{\Psi_j}\Psi_k$. Since $\langle \Psi^1, \Psi^1 \rangle = \hat{a}^2$, Ψ^1 is an isometric immersion of $S^{2p_1+1}(\hat{a})$ into $\mathbb{C}^n$, hence M_1 is an open part of $S^{2p_1+1}(\hat{a})$.

In a similar manner, we see that Ψ^2 is an isometric immersion of $S^{2p_2+1}(\hat{b})$ into $\mathbb{C}^n$, where $\hat{b} = b(a^2+b^2)^{-1/2}$ and so M_2 is an open part of $S^{2p_2+1}(\hat{b})$.

By using Proposition 3.2 in [5], M_j is contained in a real $(2p_j+2)$-dimensional totally geodesic linear subspace $\bar{M}_j$ of $\mathbb{C}^n$, for $j \in \{1,2\}$. Since M_j is not totally real in $\mathbb{C}^n$, M_j is immersed into $\mathbb{C}^n$ as a proper CR-submanifold (in the sense of Bejancu [2]) of CR-dimension p_j and hence, $\bar{M}_j$ is a holomorphic linear subspace in $\mathbb{C}^n$. Further since $\langle \Psi^1, \Psi^2 \rangle = 0$, by using an appropriate coordinate for $\mathbb{C}^n$, we may express the holomorphic linear subspace $\bar{M}_j$ as

$$\begin{aligned}\bar{M}_1 &= \{(z,0) \in \mathbb{C}^n : z \in \mathbb{C}^{p_1+1}, 0 \in \mathbb{C}^{p_2+1}\},\\ \bar{M}_2 &= \{(0,w) \in \mathbb{C}^n : 0 \in \mathbb{C}^{p_1+1}, w \in \mathbb{C}^{p_2+1}\}.\end{aligned}$$

Therefore, the immersion Ψ is given by $\Psi(r,x_1,x_2) = (rx_1, rx_2)$, where $x_1 \in S^{2p_1+1}(\hat{a})$ and $x_2 \in S^{2p_2+1}(\hat{b})$. □

Now, we consider the general case $\dim \Lambda_0 > 1$. In the rest of this section, we let $\hat{\Lambda}^\perp = \{X \in \Lambda_0 : X \perp \phi U\}$ and $\hat{\Lambda} = \mathrm{Span}\{\phi U\} \oplus \Lambda_1 \oplus \Lambda_2$.

Lemma 26 *The distributions $\hat{\Lambda}$ and $\hat{\Lambda}^\perp$ are integrable. Furthermore, if $\hat{M}$ and $\hat{M}^\perp$ is a leaf of $\hat{\Lambda}$ and $\hat{\Lambda}^\perp$ respectively, then locally, $\hat{M}^\perp$ is isometric to $\mathbb{C}^{p_0}$, and $M = \hat{M} \times \mathbb{C}^{p_0} \subset \mathbb{C}^{p_1+p_2+2} \times \mathbb{C}^{p_0}$.*

Proof For any $X, Y \in \Gamma(\hat{\Lambda}^\perp)$, by virtue of Lemma 12, we have $-A\nabla_X Y = 0$, so $\nabla_X Y \in \Gamma(\Lambda_0)$. Furthermore, Lemma 20 imply that $\nabla_X Y \perp \phi U$ and so $\nabla_X Y \in \Gamma(\hat{\Lambda}^\perp)$. This implies that the distributions $\hat{\Lambda}$ and $\hat{\Lambda}^\perp$ are integrable; and their leaves $\hat{M}$ and $\hat{M}^\perp$ are totally geodesic in M. It follows that M is locally a Riemannian product $\hat{M}^\perp \times \hat{M}$.

Since $\nabla_X Y \in \Gamma(\hat{\Lambda}^\perp)$ and $AX = 0$, for any $X, Y \in \Gamma(\hat{\Lambda}^\perp)$, we have $\bar{\nabla}_X Y \in \Gamma(\hat{\Lambda}^\perp)$, hence $\hat{M}^\perp$ can be isometrically immersed in $\mathbb{C}^n$ as a totally geodesic submanifold. Since $\hat{M}^\perp$ is invariant by J, $\hat{M}^\perp$ is locally isometric to a holomorphic linear subspace $\mathbb{C}^{p_0}$.

When we consider $\hat{M}$ as a submanifold in $\mathbb{C}^n$, the normal space of $\hat{M}$ at $x \in \hat{M}$, $T_x^\perp \hat{M} = T_x^\perp M \oplus \hat{\Lambda}_x^\perp$. $\hat{\Lambda}^\perp$ is a J-invariant subbundle of the orthogonal complement of the first normal space in $T^\perp \hat{M}$. Moreover, $\hat{\Lambda}^\perp$ is invariant under parallel translation with respect to the normal connection induced on the normal bundle $T^\perp \hat{M}$ of $\hat{M}$, by a codimension reduction theorem (cf. [9]), $\hat{M}$ can be confined in a totally geodesic holomorphic linear subspace $\mathbb{C}^{p_1+p_2+2}$ that is orthogonal to $\hat{\Lambda}^\perp$. This completes the proof. □

We are now in a position to prove Theorem 3 and Theorem 4.

Proof (of Theorem 3) It immediately follows from Theorem 7, Theorem 8 and Lemma 26. □

Proof (of Theorem 4) It immediately follows from Theorem 3, Theorem 7 and Lemma 11. □

7 A class of real hypersurfaces in $\mathbb{C}P^n$

Recall that the Euclidean metric tensor on $\mathbb{C}^{n+1}$ is given by

$$\langle z, w \rangle = \Re \sum_{k=0}^{n} z_k \bar{w}_k$$

where $z=(z_0,z_1,\cdots,z_n), w=(w_0,w_1,\cdots,w_n)\in\mathbb{C}^{n+1}$.

Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ centered at the origin. For any $z\in S^{2n+1}$, we put $\tilde{\xi}=-iz$, $\tilde{\eta}$ the 1-form on S^{2n+1} dual to $\tilde{\xi}$, and $\tilde{\phi}X=iX-\tilde{\eta}(X)z$, for any $X\in T_zS^{2n+1}$. Then $(\tilde{\phi},\tilde{\xi},\tilde{\eta})$ is a Sasakian structure on S^{2n+1} and

$$(\tilde{\nabla}_X\tilde{\phi})Y=-\tilde{\eta}(Y)X+\langle X,Y\rangle\tilde{\xi},\quad \tilde{\nabla}_X\tilde{\xi}=-\tilde{\phi}X$$

for any $X,Y\in\Gamma(TS^{2n+1})$, where $\tilde{\nabla}$ is the Levi-Civita connection on S^{2n+1} induced by the Euclidean metric tensor of $\mathbb{C}^{n+1}$.

Suppose M' is a real hypersurface in S^{2n+1} tangent to $\tilde{\xi}$ and N' a unit vector field normal to M'. Putting $\xi=-\tilde{\phi}N'$, η' the 1-form on M' dual to ξ' and $\phi'X=\tilde{\phi}X-\eta'(X)N'$, we have

$$X=-\phi'^2X+\eta'(X)\xi'+\tilde{\eta}(X)\tilde{\xi}$$

for any X tangent to M'. Let ∇' and A' be the induced Levi-Civita connection and shape operator on M' respectively. Then we have

$$\begin{aligned}\nabla'_X\tilde{\xi}&=-\phi'X,\quad A'\tilde{\xi}=-\xi' \\ (\nabla'_X\phi')Y&=\eta'(Y)A'X-\langle A'X,Y\rangle\xi'-\tilde{\eta}(Y)X+\langle X,Y\rangle\tilde{\xi},\quad \nabla'_X\xi'=\phi'A'X\end{aligned}\tag{25}$$

for any X,Y tangent to M'.

The unit circle S^1 acts freely on S^{2n+1}, i.e., $(z,\lambda)\in S^{2n+1}\times S^1\mapsto z\lambda\in S^{2n+1}$. Under the identification induced by the action, the orbit space S^{2n+1}/S^1 is a complex projective space $\mathbb{C}P^n$ of constant holomorphic sectional curvature 4, i.e., $c=1$.

Denote by $\pi:S^{2n+1}\to\mathbb{C}P^n$ the Hopf fibration. The vertical subspace $\mathscr{V}_z=\mathrm{Span}\{\tilde{\xi}\}$ at each $z\in S^{2n+1}$, and the horizontal subspace $\mathscr{H}_z=\{w\in\mathbb{C}^{n+1}:\langle z,w\rangle=\langle iz,w\rangle=0\}$ is invariant by the action of S^1. Furthermore, the canonical projection π induced a linear isomorphism of $\mathscr{H}_z$ onto $T_{\pi(z)}\mathbb{C}P^n$, hence, up to identification, we have the following decomposition

$$T_zS^{2n+1}=T_{\pi(z)}\mathbb{C}P^n\oplus\mathrm{Span}\{\tilde{\xi}\}.$$

Denote by J the induced complex structure, $\langle,\rangle$ the Riemannian metric tensor and $\bar{\nabla}$ the Levi-Civita connection of $\mathbb{C}P^n$ respectively. Then we have the following identities:

$$\begin{aligned}(JX)^H&=\tilde{\phi}X^H=-\tilde{\nabla}_{X^H}\tilde{\xi}=-\tilde{\nabla}_{\tilde{\xi}}X^H \\ (\langle X,Y\rangle)^H&=\langle X^H,Y^H\rangle \\ (\bar{\nabla}_XY)^H&=\tilde{\nabla}_{X^H}Y^H-\langle\tilde{\phi}X^H,Y^H\rangle\tilde{\xi}\end{aligned}$$

for any X,Y tangent to $\mathbb{C}P^n$, where X^H denotes the horizontal lift of X, etc.

Let $\Psi:M\to\mathbb{C}P^n$ be an isometric immersion. Then $M'=\pi^{-1}(M)$ is a principal S^1-bundle over M with totally geodesic fibers. The lift $\Psi':M'\to S^{2n+1}$ is an isometric immersion so that $\pi\circ\Psi'=\Psi\circ\pi$, i.e.,

$$\begin{array}{ccc} M' & \xrightarrow{\Psi'} & S^{2n+1} \\ \pi\downarrow & & \downarrow\pi \\ M & \xrightarrow{\Psi} & \mathbb{C}P^n \end{array}\tag{26}$$

Conversely, for a S^1-invariant isometric immersion $\Psi':M'\to S^{2n+1}$, there is a unique isometric immersion $\Psi:M\to\mathbb{C}P^n$ such that $\pi\circ\Psi'=\Psi\circ\pi$.

Now consider real hypersurfaces M and $\hat{M}$ respectively in $\mathbb{C}P^n$ and S^{2n+1} as described above such that the diagram (26) commutes. From the above identities, we obtain

$$\begin{aligned} N^H &= N', \quad \xi^H = \xi', \quad (\phi X)^H = \phi' X^H, \\ (\nabla_X Y)^H &= \nabla'_{X^H} Y^H - \langle \phi' X^H, Y^H \rangle \tilde{\xi}, \\ (AX)^H &= A' X^H + \langle X^H, \xi' \rangle \tilde{\xi} \end{aligned} \tag{27}$$

for any X, Y tangent to M.

Consider the invariant distribution $\mathscr{D}'$ on M' given by $\mathscr{D}'_w = \{X^H : X \in \mathscr{D}_{\pi(w)}\}$, $w \in M'$. Now for any $X, Y \in \Gamma(\mathscr{D})$, we see that

$$\begin{aligned} \langle (\phi' A' - A' \phi') X^H, Y^H \rangle &= \langle \phi' (AX)^H - A' (\phi X)^H, Y^H \rangle \\ &= \langle (\phi AX)^H - (A \phi X)^H, Y^H \rangle \\ &= (\langle (\phi A - A \phi) X, Y \rangle)^H. \end{aligned}$$

Since π_* is a linear isomorphism of $\mathscr{D}'_w$ to $\mathscr{D}_{\pi(w)}$, $w \in M'$, we have proved the following lemma.

Lemma 27 *Let M and M' be real hypersurfaces, respectively in $\mathbb{C}P^n$ and S^{2n+1} as described above. Suppose the diagram (26) commutes. Then M satisfies (2) if and only if M' satisfies*

$$\langle (\phi' A' - A' \phi') X', Y' \rangle = 0 \tag{28}$$

for any $X', Y' \in \Gamma(\mathscr{D}')$.

Assume that $\operatorname{grad} \alpha = \alpha V$. For any $X \in \Gamma(TM)$, we have

$$X^H \alpha' = (X\alpha)^H = (\alpha \langle X, V \rangle)^H = \alpha' \langle X^H, V^H \rangle$$

where $\alpha' = \alpha \circ \pi$. Next since $\tilde{\xi} \alpha' = (\pi_* \tilde{\xi}) \alpha = 0$, we conclude that

$$\operatorname{grad} \alpha' = \alpha' V^H.$$

Conversely, it is clear that the above condition implies $\operatorname{grad} \alpha = \alpha V$. Hence we have

Lemma 28 *Let M and M' be real hypersurfaces, respectively in $\mathbb{C}P^n$ and S^{2n+1} as described above. Then* $\operatorname{grad} \alpha = \alpha V$ *if and only if* $\operatorname{grad} \alpha' = \alpha' V^H$.

We may also, alternatively, regard $\mathbb{C}P^n$ as the collection of equivalent classes of complex lines in $\mathbb{C}^{n+1}_* = \mathbb{C}^{n+1} - \{0\}$. The group $\mathbb{C}_* = GL_1(\mathbb{C})$ acts freely on $\mathbb{C}^{n+1}_*$, i.e., $(z, \lambda) \in \mathbb{C}^{n+1}_* \times \mathbb{C}_* \mapsto z\lambda \in \mathbb{C}^{n+1}_*$. Under the identification induced by the action, the orbit space $\mathbb{C}^{n+1}_* / \mathbb{C}_*$ is $\mathbb{C}P^n$.

Denote by $\psi : \mathbb{C}^{n+1}_* \to \mathbb{C}P^n$ the cannonical projection. Suppose $\Psi : M \to \mathbb{C}P^n$ is an isometric immersion. Then $\hat{M} = \psi^{-1}(M)$ is C_*-invariant and it is a principal $\mathbb{C}_*$-bundle over M. Further the lift $\hat{\Psi} : \hat{M} \to \mathbb{C}^{n+1}_*$ is an isometric immersion so that $\psi \circ \hat{\Psi} = \Psi \circ \psi$, i.e.,

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\hat{\Psi}} & \mathbb{C}^{n+1}_* \\ \psi \downarrow & & \psi \downarrow \\ M & \xrightarrow{\Psi} & \mathbb{C}P^n \end{array} \tag{29}$$

Conversely, for a $\mathbb{C}_*$-invariant isometric immersion $\hat{\Psi}: \hat{M} \to \mathbb{C}_*^{n+1}$, there is a unique isometric immersion $\Psi: M \to \mathbb{C}P^n$ such that $\pi \circ \Psi' = \Psi \circ \pi$.

With the above notation, $\hat{M}$ is the *cone over* M' with the origin of $\mathbb{C}^{n+1}$. $\hat{M}$ is a warped product of $\mathbb{R}_* \times_r M'$, where $\mathbb{R}_* = \mathbb{R} - \{0\}$ and r a coordinate of $\mathbb{R}_*$. The immersions Ψ' : $M' \to S^{2n+1}$ and $\hat{\Psi}: \hat{M} \to \mathbb{C}_*^{n+1}$ are related by

$$\hat{\Psi}(r,x) = r\Psi'(x), \quad \forall r \in \mathbb{R}_*, \, x \in M'. \tag{30}$$

For a vector X tangent to M', we denote by the same X, the extension of X along the rays of the cone $\hat{M}$ over M' by parallel translation, and denote by the same $\mathscr{D}'$, the distribution of all such vector fields in $\hat{M}$. Further, we denote by $\hat{A}$ the shape operator of $\hat{M}$, $(\hat{\phi}, \hat{\xi}, \hat{\eta})$ the induced almost contact structure on $\hat{M}$, etc. It follows from (30) that

$$\tilde{\xi} = -\hat{\phi}(\partial\hat{\Psi}/\partial r), \quad \hat{\xi} = \xi', \quad \hat{N} = N', \tag{31}$$

$$\hat{\phi}X = \phi' X, \quad \forall X \in \Gamma(\mathscr{D}'),$$

$$\hat{A}X_{(r,x)} = \frac{1}{r}A'X_x, \quad \forall r \in \mathbb{R}_*, \, x \in M', \tag{32}$$

$$\hat{A}\hat{\phi}\tilde{\xi} = 0.$$

Also, if f is a real-valued function on M' and $\hat{f}$ a real-valued function on $\hat{M}$ given by $\hat{f}(r,x) = f(x)$, $r \in \mathbb{R}_*$, $x \in M'$ then from (30), we have

$$X_{(r,x)}\hat{f} = \frac{1}{r}X_x f. \tag{33}$$

The holomorphic distribution $\hat{\mathscr{D}}$ on $\hat{M}$ is given by

$$\hat{\mathscr{D}} = \mathscr{D}' \oplus \operatorname{Span}\{\tilde{\xi}, \hat{\phi}\tilde{\xi}\}.$$

Lemma 29 *Let M and $\hat{M}$ be real hypersurfaces, respectively in $\mathbb{C}P^n$ and $\mathbb{C}_*^{n+1}$ as described above. Suppose the diagram (29) commutes. Then M satisfies (2) if and only if $\hat{M}$ satisfies*

$$\langle(\hat{\phi}\hat{A} - \hat{A}\hat{\phi})X, Y\rangle = 0 \tag{34}$$

for any $X, Y \in \Gamma(\hat{\mathscr{D}})$.

Proof Let M' be a real hypersurface S^{2n+1} such that the diagram (26) commutes. Note that the equation (28) is equivalent to

$$(\phi' A' - A'\phi')X' = -\langle A'\phi' X', \xi'\rangle\xi' \tag{35}$$

for any $X \in \Gamma(\mathscr{D}')$. Also, the equation (34) is equivalent to

$$(\hat{\phi}\hat{A} - \hat{A}\hat{\phi})X = -\langle\hat{A}\hat{\phi}X, \hat{\xi}\rangle\hat{\xi} \tag{36}$$

for any $X \in \Gamma(\hat{\mathscr{D}})$.

(35) $\Rightarrow$ (36). If (35) holds, by a direct calculation,

$$(\hat{\phi}\hat{A} - \hat{A}\hat{\phi})\tilde{\xi} + \langle\hat{A}\hat{\phi}\tilde{\xi}, \hat{\xi}\rangle\hat{\xi} = 0$$

$$(\hat{\phi}\hat{A} - \hat{A}\hat{\phi})\hat{\phi}\tilde{\xi} - \langle\hat{A}\tilde{\xi}, \hat{\xi}\rangle\hat{\xi} = 0$$

$$(\hat{\phi}\hat{A} - \hat{A}\hat{\phi})X + \langle\hat{A}\hat{\phi}X, \hat{\xi}\rangle\hat{\xi} = \frac{\phi' A' X - A'\phi' X}{r} + \frac{\langle A'\phi' X, \hat{\xi}\rangle\hat{\xi}}{r} = 0$$

for any $X \in \Gamma(\mathscr{D}')$. This proved (36).

(36) $\Rightarrow$ (35). This can be done using a similar argument as in the derivatiion of the previous equation.

Hence, the lemma follows from Lemma 27. □

By (27), we see that

$$A'\xi' = \alpha'\xi' - \phi' V' - \tilde{\xi}.$$

Hence

$$\hat{A}\hat{\xi} = \hat{\alpha}\hat{\xi} + \hat{\beta}\hat{U}$$

where $\hat{\alpha} = \alpha'/r$, $\hat{\beta} = \sqrt{\beta'^2+1}/r$, $\hat{U} = -(\hat{\phi}V' + \tilde{\xi})/\sqrt{\beta'^2+1}$.

Now suppose $\operatorname{grad}\alpha' = \alpha' V'$. Since $\hat{U}$ is tangent to M' and $\hat{U} \perp V'$, from the hypothesis we have $\hat{U}\hat{\alpha} = 0$. Also, for X tangent to M' with $X \perp \hat{U}, \hat{\phi}\hat{U}$, we have $X\hat{\alpha} = 0$. Finally, it follows from (31) that

$$\hat{\phi}\hat{U} = \frac{V' - \partial\hat{\Psi}/\partial r}{\sqrt{\beta'^2+1}}.$$

By using (33) and the assumption $\operatorname{grad}\alpha' = \alpha' V'$, we have

$$\hat{\phi}\hat{U}\,\hat{\alpha} = \frac{\alpha'\beta'^2 + \alpha'}{r^2\sqrt{\beta'^2+1}} = \hat{\alpha}\hat{\beta}.$$

Hence, we conclude that $\operatorname{grad}\hat{\alpha} = \hat{\alpha}\hat{\beta}\hat{\phi}\hat{U}$. Conversely, from a similar calculation, we see that $\operatorname{grad}\hat{\alpha} = \hat{\alpha}\hat{\beta}\hat{\phi}\hat{U}$ implies $\operatorname{grad}\alpha' = \alpha' V'$. According to Lemma 28, we obtain

Lemma 30 *Let M and $\hat{M}$ be real hypersurfaces, respectively in $\mathbb{C}P^n$ and $\mathbb{C}_*^{n+1}$ as described above. Suppose the diagram (29) commutes. Then* $\operatorname{grad}\alpha = \alpha V$ *if and only if* $\operatorname{grad}\hat{\alpha} = \hat{\alpha}\hat{\beta}\hat{\phi}\hat{U}$.

Proof (of Theorem 5) It is clear that the real hypersurface $\hat{M}$ in $\mathbb{C}_*^{n+1}$ consisting of points $w = (rx_1, rx_2, z)$ is invariant under the action of $\mathbb{C}_*$. Hence the necessity part follows from Lemma 29, Lemma 30 and Theorem 3.

Conversely, assume that M is a real hypersurface in $\mathbb{C}P^n$, $n \geq 3$, with $\operatorname{grad}\alpha = \alpha V$ and satisfying the condition (2).

Let G_0 be the open set consisting of points x for which $\beta(x) \neq 0$. If G_0 is empty then ξ is principal and $\phi A = A\phi$, so by Theorem 1, M is of type A_1 or A_2. From the construction of these real hypersurfaces (cf. [25]), under the projection ψ, M is the image of $\hat{M}$, a cone over the clifford hypersurface $S^{2p_1+1} \times S^{2p_2+1}$ in S^{2n+1}. Hence M is defined by the immersion (3) with $p_0 = 0$.

Now suppose G_0 is non-empty. Then it follows from Lemma 16 that M has at least four and at most five distinct principal curvatures λ_j, $j \in \{0,1,2,3,4\}$ in G_0, where

$$\lambda_0 = 0; \quad \lambda_1, \lambda_2 = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta^2}}{2}; \quad \lambda_3, \lambda_4 = \frac{\alpha \pm \sqrt{\alpha^2 + 4(1+\beta^2)}}{2} \tag{37}$$

with constant multiplicities $2p_0+1$, 1, 1, $2p_1$ and $2p_2$ respectively, on G_0. Here we have considered $c = 1$.

We shall show that G_0 is a dense subset in M. Suppose to the contrary that the interior of $M - G_0$, $\operatorname{Int}(M - G_0)$ is non-empty. Note that the open submanifold $\operatorname{Int}(M - G_0)$ is of type A_1 or A_2, hence, it has at least two and at most three locally constant principal curvatures α, μ_1 and μ_2, with multiplicities 1, $2(n-q-2)$ and $2q$ respectively on $\operatorname{Int}(M - G_0)$. In particular, μ_1 and μ_2 are locally nonzero constant on $\mathrm{Int}(M-G_0)$ (cf. [25]). At the boundary points of G_0, it follows from (37) that the principal curvature $\lambda=0$ has multiplicity $2p_0+3$. But this contradicts the fact that μ_1 and μ_2 are locally constant in $\mathrm{Int}(M-G_0)$ and the continuity of the principal curvatures. Hence, we have showed that G_0 is open and dense in M. Accordingly, up to rigid motions, M is defined by the immersion (3) with $p_0>0$. This completes the proof. □

Proof (of Theorem 6) Note that M is minimal if and only if $\hat{M}$ is minimal. Hence, by Theorem 4 and Theorem 5, we obtain the classification of minimal real hypersurfaces in $\mathbb{C}P^n$ satisying condition (2). □

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