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Health Monitoring of Cantilever Rod Using Vibration Test “Theoretical and Numerical Study “

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The aim of this study is to assess the numerical analysis by using a theoretical model for detecting cracks on cantilever rod. A theoretical model for the cantilever rod was based on the Euler–Bernoulli beam theory. The theoretical model was used to calculate the natural frequency for the uncracked rod, as well as for the cracked rod. The effect of the crack on the natural frequencies was inserted by using the second moment of inertia for the cracked cross section, where it influenced the natural frequency and led to decrease the stiffness of the cracked beam. The theoretical results showed that the reduction in natural frequencies (RFI) was a good indicator for detecting the damage magnitude. A finite element model for a cantilever solid rod was created. The Eigen analysis was conducted to determine the dynamic properties (natural frequency and mode shape). The crack detection was performed by monitoring the changes in the first four natural frequencies. It is hypothesized that as the natural frequency of a structure is directly corresponded to its stiffness, the monitoring of any changes exhibited in the natural frequency can lead to the detection in the change of the stiffness of the structure. The mass was assumed constant because the amount of lost due to crack was very less, hence it was neglected. The theoretical model and natural frequency’s results were in good agreement with numerical results.

1. Introduction

A number of non-destructive techniques are available to detect faults and defects in a structure. Their use and reliability have consequently become an important factor in the evaluation of the safety of any structure, Owolabi et al. [1]. Various engineers and scientists have devoted their time and efforts towards developing a new, more reliable, efficient, and less tedious detection techniques as non destructive evaluation technique such as magnetic particles and ultrasonic waves. A study on the free and forced vibration was presented by Yang et al., [2]. In this study, the authors studied the beam containing an open edge cracks on a homogeneous Euler-Bernoulli beams. Karthikeyana et al., [3] developed a measurement method for the crack localization and sizing in a beam from a free and forced response. The beam modelling for transverse vibrations was Timoshenko beam theory. Free and forced vibration for beam's cracked analysis using finite element method (FEM). Kisa and Gurel, [4] analyzed the free vibration for uniform and step cracked beams with circular cross section and present a novel numerical technique. In this technique, the finite element and component mode synthesis methods are used together. The authors explained the important aspect to assess failure of structure based on the knowledge and information of cracked beams. In addition, they found that locations and sizes of crack could particularly affect the modal features, i.e. when the location of cracks at the step part of beams, the effect will be clearly on dynamic properties.

Viola et al. [5] investigated the changes in the magnitude of natural frequencies and modal response with the presence of a crack on an axially loaded uniform using Timoshenko beam theory. It was noticed that the larger crack has more significant effect on the frequency when cracks of different depths are considered in this study. Kisa and Gurel [6] proposed a numerical model that combines the finite element and component mode synthesis methods for the modal analysis of beams with circular cross section and containing multiple non-propagating open cracks. This study investigated the influence of the location and the depth of crack on the dynamic properties (mode shape and frequencies) for beams Zheng and Kessissoglou[7] used the finite element method to obtain the dynamic properties (natural frequencies and mode shape) of cracked beam. To calculate the vibration modes for cracked beam, the authors developed a new shape function. This function was satisfied for the local flexibility conditions at the crack locations. A new stiffness matrix using the overall additional flexibility was found in this study. Stiffness matrix confirmed more accurate natural frequencies than using the local flexibility matrix.

A method for prediction of size and location of multiple cracks based on measurement of natural frequencies had been verified theoretically and experimentally by Patil and Maiti [8] . This method was in good accuracy to predict the location of crack and lowers for prediction of the size of crack. In addition, the accuracy decrease when crack's number increase. Loutridisa et al., [9] proposed a new method for crack detection in beams based on instantaneous frequency and empirical mode decomposition. They found that the instantaneous frequency was a good indicator for sizing of crack. The crack detection in cantilever beams by vibration analysis was established by Nahvi and Jabbari [10]. The results of an experimental and theoretical analysis found that the position and size of the crack affected on the first and second natural frequency of beam with cracked beam. Leontios, et al. [11] presented a new technique for damage detection (crack) in beam structures based on the kurtosis crack detector (KCD). Based on the analysis result for the fundamental vibration mode of the cracked cantilever beam for both location and size of the crack were estimated. Kisa, [12] presented an investigation for the crack effects on the cantilever composite beam on the dynamic properties. Component mode synthesis and finite element were used together in this research. The influence of location, depth of crack and orientation, volume fraction of the fiber on the dynamic properties of the beams with cracks were investigated in this study. The change and drop in the dynamic properties were used to present a sufficient method to analyse the vibration in cracked cantilever composite beams, as well as can detect the damage. Multi cracks in beam detected by frequencies formulated in a non-linear optimization problem formulation, using MATLAB function, Khiem and Lien [13]. For the frequency equation, the crack was simulated as a spring model depended on the dynamic stiffness of beam with crack. The set parameters of crack to be predicted involve depth and location as well as the probable cracks. The higher accuracy to detect the crack location when there is more measurements for natural frequency. This case is not suitable to detect the depth of crack.

A simple method for determining the depth and location of cracks in double-cracked beams was proposed by Douka et al. [14]. The study included experimental and analytical investigation. The present crack caused a significant change in the anti-resonance frequencies. Thus, anti-resonance changes gave extra details about the crack appearance which balance the change of natural frequency. The experimental work of the effects of cracks and damages on the integrity of structures, with a view to detect, quantify, and determine their extents and locations were investigated by Owolabi, et al. [1]. They found that the vibration behavior of the beams was influenced by the depth and location of crack and the number of mode. Experimental and analytical investigations by Douka et al., [15] showed that the identification of crack in cantilever beam depends on wavelet analysis. Due to the rapid change in the spatial difference of the response, the crack location was determined. The intensity factor is important to estimate the crack depth. This factor relates the crack size to wavelet coefficient. Mermertas and Erol, [16] studied the transverse vibration of a cracked cantilever beams under the effect of mass attachment. They conducted theoretical investigation on the cracked beam based on the Bernoulli-Euler equation to govern the free vibrations of cracked beam elements. The effect of cracks decreased in beams presented a mass attachment in different locations.

2. CASE STUDY

In order to demonstrate the assessment of the numerical analysis based on the theoretical model, a steel sold rod was examined. The rod was fixed from one end and free from the other as cantilever. The rod length is 1 m, and rod diameter is 50 mm. the rod was assumed to be steel. The result carried out from theoretical model was used to compare with the numerical analysis by finite element modeling. This comparison was between two cases for cantilever beams, control beam without crack and beam with crack closed to the fixed end with different depths. The crack depth started at 5 mm and then increased cumulatively when it reached 30 mm crack depth. The beam with a transverse edge crack was clamped at right end, free at left end and it was a uniform structure with a constant circular cross-section. The Euler–Bernoulli beam model was assumed. The crack was assumed to be an open crack and the damping had not been considered in this study. Figure 1 illustrates the cantilever rod and shows the crack on its surface.

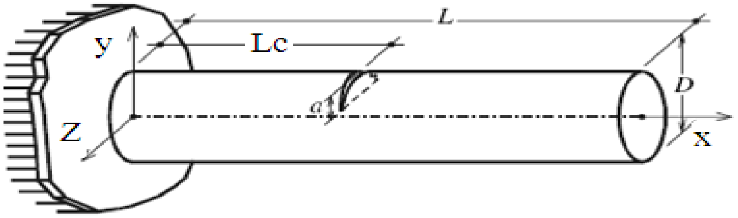

Figure 1: Cantilever solid rod with cracks

3. Theoretical Model

The theoretical model was presented to compute the change in the natural frequencies based on demonstration of a transverse crack in a rod, and this was done by representing the reduction of the second moment of inertia (*ΔI*) of the element at the location of the crack on the steel rod. The present of a crack in beam led to the reduction of its second moment of inertia which led to decrease the stiffness of the beam, and this influence the natural frequency of the original beam, Sinou [17] .

The theoretical model based on the equations of motion for the elastic beam was developed based on Euler–Bernoulli theory (Inman [18]).

$$\mathrm{E\,I}\,\frac{\partial^4 \mathrm{y}}{\partial \mathrm{x}^4} + \rho \mathrm{A}\,\frac{\partial^2 \mathrm{y}}{\partial \mathrm{t}^2} = 0 \tag{1}$$

The normal mode shape can be expressed as Equation (2):

$$\mathrm{y}_{(\mathrm{x,t})} = \mathrm{Y(x).\,e^{iwt}} \tag{2}$$

Substitute Equation (2) in Equation (1):

$$\mathrm{EI}\beta_{\mathrm{n}}{}^4.\,\mathrm{Y(x).\,e^{iwt}} - \rho \mathrm{Aw}^2.\,\mathrm{Y(x).\,e^{iwt}} = 0 \tag{3}$$

After applying the boundary condition of cantilever beam and arrangement for all the parameters in Equation (3), the final equation to calculate theoretical values of natural frequencies were formulated in the Equation (4) ,(Inman [18]).

$$\mathrm{f_n} = \frac{\beta_{\mathrm{n}}{}^2}{2\pi}\sqrt{\frac{\mathrm{EI}}{\rho \mathrm{A}}} \tag{4}$$

Where $\mathrm{f_n}$ is the natural frequency

The natural frequency calculation based on the reduction moment of inertia (*ΔI*) of the element at the location of the crack. Using equation (4) to get the result for the natural frequency depends on different parameters such as:

(β_n) Factor related to the number of mode shape

(*E*) Modules of elasticity, and

(ρ) The density per unit length

(A) cross section area

(I) second moment of inertia

The values of β_n, *E* and ρ were not affected by the crack statues. The cross sectional area *A* was effected only when the crack formed transverse the length of the beam. The effect of (*A*) value on the global natural frequency was very small, therefore it can be neglected.

As described above, the parameters (β_n, *E*, ρ and *A*) did not have any significant effect on the natural frequency. Therefore, the second moment of inertia has the major effect. The frequency change was considered to detect the damage magnitude by using theoretical approach, where any change in frequency indicate the change in second moment of inertia which can be used to find the crack depth. In this section, the breathing mechanism and modelling of the crack was discussed briefly. The change in second moment of inertia **(**I**)** was calculated for different crack depth which gave effect on the cross sectional shape as the following:

3.1 Un- Cracked Section with *a* = 0

For cracked section, the cross sectional shape depends on the crack depth (a)**.** For control case where there is no crack, the cross section is circular (a = zero). The second moment of inertia is shown in Equation (7) (Efunda, [19]):

$$I_c = \frac{\pi R^4}{4} \tag{7}$$

where

R : cross section diameter

3.2 Cracked Section with $0 < a > R$

The second moment of inertia of the cross section shape (illustrated in Figure 2) was calculated as shown in Equation 8. The segment parameters are shown in Figure 3.

$$I_{cr} = \left[(I_c + A_c \, x \, (y^{\circ})^2) - (I_a + A_a \, x \, (y^{\circ} + y_a)^2) + (I_t + A_t \, x \, (y^{\circ} + 2/3h)^2)\right] \tag{8}$$

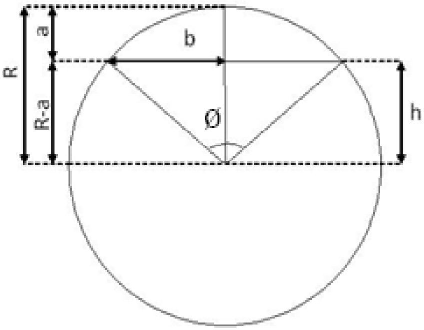

Figure 2: Cross section for beam with crack

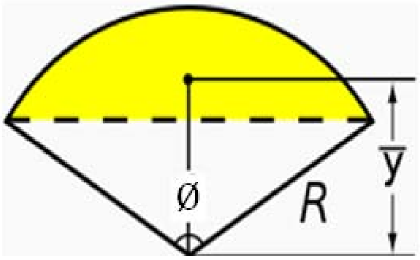

Figure 3: Area of the arch and segment

Where :

$$A_c = \pi R^2$$

$$I_a = \left(\emptyset - \frac{1}{2} \mathrm{SIN} 2\emptyset\right) \frac{R^4}{8}$$

$$A_a = A_c \, x \frac{\emptyset}{360}$$

$$y^{\circ} = R - y'$$

$$y' = \frac{(A_c \, x \, R) - [A_a \, x \, (y_a + R)] + [(A_t \, x (y_t + R)]}{(A_c - A_a + A_t)}$$

$$\mathrm{A_t} = \frac{1}{2}\mathrm{bh}$$

$$\mathrm{y_t} = \frac{2}{3}\mathrm{h}$$

$$\mathrm{y_a} = \frac{4\mathrm{R}}{3}\left[\frac{\sin^3\frac{\emptyset}{2}}{(\emptyset - \sin\emptyset)}\right]$$

$$\mathrm{I_t} = \frac{\mathrm{bh}^3}{36}$$

And :

I_{cr}	second moment of inertia for cracked section
I_c	second moment of inertia for circular section
A_c	area of circular section
I_a	second moment of inertia for the arch
A_a	area of the arch
y°	different between the untracked sections centred and cracked section centered
y'	centre of gravity of whole section
A_t	area of triangular
y_t	centroid of the triangular from the centre of the circle
y_a	centroid of the Arch from the centre of the circle
I_t	second moment of inertia for triangular
h	radius of circular subtract depth of crack

3.3 Cracked Section with *a = R*

Second moment of inertia was calculated for half circle cross section as following

$$0.5\ \mathrm{I_{circle}} = \frac{1}{2}\frac{\pi \mathrm{R}^4}{4} \qquad (9)$$

To transfer the location of the new centered of the half circle to the location of distance equal to $(\frac{4R}{3\pi})$ as shown in Equation 10:

$$\mathrm{I_{crack}} = \mathrm{I_{1/2circle}} + \mathrm{A_{1/2circle}}\ \mathrm{x}\ (\frac{4\mathrm{R}}{3\pi})^2 \qquad (10)$$

3.4 Cracked Section with *a > R*

The centroid of the segment is Y_{seg} and calculated as in Equation (11).

$$\mathrm{y_{seg}} = \frac{4\mathrm{R}}{3}\left[\frac{\sin^3\frac{\emptyset}{2}}{(\emptyset - \sin\emptyset)}\right] \qquad (11)$$

$$\mathrm{A_{seg}} = \frac{\mathrm{R}^2}{2}(\emptyset - \sin\emptyset) \qquad (12)$$

A_{seg}: area of the segment as shown in Figure 3 and calculated as Equation (12).
The center of the un-cracked area calculated as Equation (13):

$$y' = \frac{(A_{seg} x\, y_{seg} + A_t x\, y_t)}{(A_{seg} + A_t)} \quad (13)$$

$$I_{crack} = \left[I_a + A_a x\, (y')^2\right] + [I_t + A_t x\, (y' - y_t)^2] \quad (14)$$

4. Numerical analysis (Finite Element Model)

A solid steel rod was modelled using ANSYS Finite Element Software. The rod was modelled with different crack depth. A crack depth started from 5 mm up to 30 mm. The crack effects on the dynamic properties were applied with all pervious properties mentioned in this session. The suitable element was selected as SOLID95, 20-node 95 from [Library of Element Type] ANSYS has characteristic in dealing with two and three dimension; it can tolerate irregular shape and keeping the same accuracy.

Appropriate type of metal depending on the applications and the purpose of the study was chosen to present successful study and to get scientific value. In this study, steel rod was used with properties:

$E = 216 \times 10^3$ N/mm^2,
$\rho = 7.85 \times 10^{-6}$ Kg/mm^3 and
$\upsilon = 0.33$

After creating the mesh for the beam, the boundary condition was chosen for the cantilever beam model and its boundary was fixed at one end only. Figure 4 illustrates the finite element model of the beam.

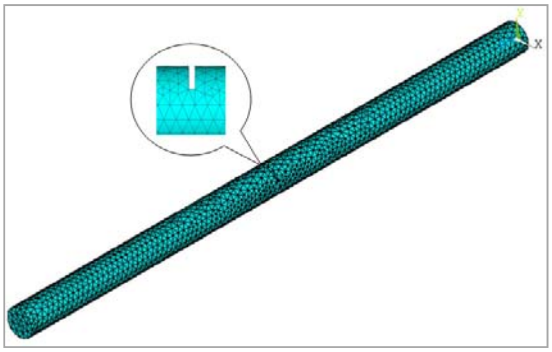

Figure 4: Meshing of Cantilever Steel Beam

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5. Results and discussion

This section shows the theoretical and numerical results as well as the comparison between both of them. The comparison between numerical and theoretical results was presented in term of the different between them based on the numerical result according to Equation (15)

$$\text{Discrepancy} = \left|\frac{F_T - F_N}{F_N}\right| .100\% \quad (15)$$

where : F_T is the theoretical frequency and F_N is the numerical frequency.

The assessment was evaluated by calculating the differences in values for natural frequency between numerical and theoretical results. Table 1 and 2 tabulated the numerical and theoretical model for the cantilever beam with controlled and damaged cases. The controlled case represented (no crack), whereas, the damaged cases had crack depth at the quarter span.

Table 1: The Comparison between Numerical and Theoretical Results for the First Four Natural Frequencies of Un-Cracked Cantilever Beam

Control			
Natural Frequency	Numerical	Theoretical	Discrepancy %
W1	32.7	32.7	0.0
W2	228.3	230	0.2
W3	631.0	645	2.1
W4	1214	1260	3.6

Table 2: The Comparison between Numerical and Theoretical Results for the First Four Natural Frequencies of the Cracked Cantilever Beam with Different Depth of Crack

	a/D = 0.1			a/D = 0.2		
Natural Frequency	**Numerical**	**Theoretical**	**Discrepancy%**	**Numerical**	**Theoretical**	**Discrepancy%**
W1	32.6	32.1	0.0	32.4	31.8	1.8
W2	228.3	226	1.0	228.3	216	5.6
W3	630.4	633	0.4	627.6	599	4.7
W4	1212.7	1240	2.2	1206.6	1140	5.8
	a/D = 0.3			a/D = 0.4		
Natural Frequency	**Numerical**	**Theoretical**	**Discrepancy%**	**Numerical**	**Theoretical**	**Discrepancy%**
W1	31.5	30.6	2.9	31.1	29.8	4.3
W2	228.1	215	6.0	228.0	211.2	7.9
W3	621.0	587	5.7	610.8	568	7.5
W4	1191.9	1123	6.1	1167.7	1110	5.1
	a/D = 0.5			**a/D = 0.6**		
Natural Frequency	**Numerical**	**Theoretical**	**Discrepancy%**	**Numerical**	**Theoretical**	**Discrepancy %**
W1	29.4	31	5.1	27.6	30	8.0
W2	226.2	209.8	7.8	223.3	206.8	7.9
W3	591.2	571	3.5	565.1	587	5.2
W4	1120.4	1170	4.2	1055.7	1130	6.5

The different between the numerical and theoretical frequencies were also presented as dimensionless values as shown in Equation 16

$$\text{Dimentionless frequency comaprision } = \frac{\mathrm{F}_T}{\mathrm{F}_N} \qquad (16)$$

Figure 5 shows the comparison between numerical and theoretical results for different crack depths. The frequencies of the cracked beam were lower than frequencies of corresponding intact beam**.** The differences in the natural frequencies increased with increase of the depth of crack, because the reduction in second moment of inertia led to reduce the stiffness (Kisa and Gure [4], and Sinou [17] and Lees [19]). The comparison between numerical and theoretical results was less than 9 %, which is acceptable.

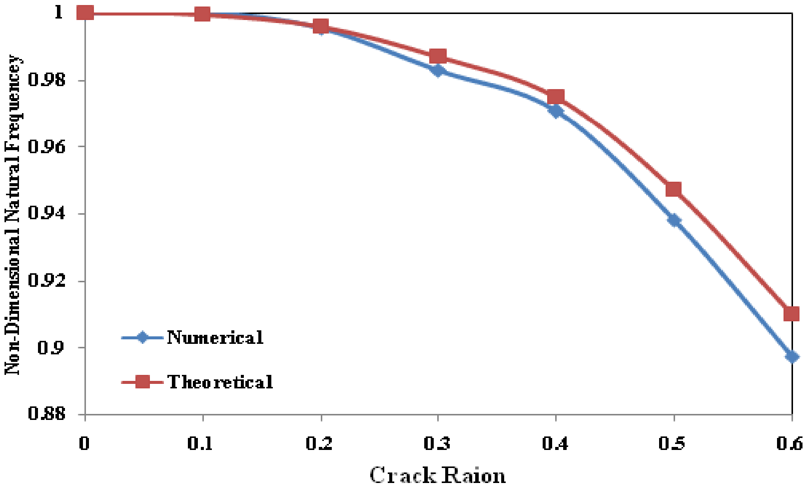

Figure 5: Non-dimensional natural frequencies against crack depth

The natural frequencies of cracked cantilever beam were lower than the un-cracked beam. In contrast, when the crack's depth increased, the differences in the frequencies increased too. This is consistent with Kisa and Gurel [4] & [6], in which similarly to the findings.

The presence of curvature of the mode shape in different location on the beam was based on the number of mode shape, where the curvature increased with the presents of peaks on the mode shape. This shows that change in the dynamic's properties values were due to change in the curvature portion (Owolabi et al., [1]). The shape of the modes depended on the mode numbers as well as the boundary condition of the beam

Cantilever beams have distinctive mode shapes. The result of the first four natural frequencies and mode shapes were investigated. The curvature of the mode shape was influenced by the changing in dynamic properties (natural frequency and mode shape) and, the severity of the damage was computed by monitoring the change in the natural frequency used the (FRI).

Frequency Reduction Index (FRI) depended on the differences in the natural frequency that occurred due to the crack. In this study, the crack influenced the natural frequency as explained and the reduction in the values related to all the crack's depth. The determination of FRI was given in Equation 17.

$$\mathrm{FRI} = \frac{f_d - f_c}{f_c} * 100\% \tag{17}$$

where:

f : is the frequency , c : is indicating for control case and d : indicate for damage case

Figure 6 shows the FRI of the cracked beam against the crack depth ratio (*a/D*) for first four modes for the numerical results. The FRI values decreased when the depth of the crack increased.

This reduction of FRI was, probably due to the effect of the crack on the second moment of inertia (*ΔI*) of the element, and the position of the crack on the beam. As the second moment of inertia for circular section was affected by the beam stiffness, consequently influenced the natural frequency (Sinou [17]). For small depth of crack, the effect on the natural frequency was not obvious for all modes. This may indicate that the FRI is, probably, not a suitable tool to predict the small crack size. With the increase in the depth of the crack, the FRI increases for all the measured modes and it shows high sensitivity to the crack depth.

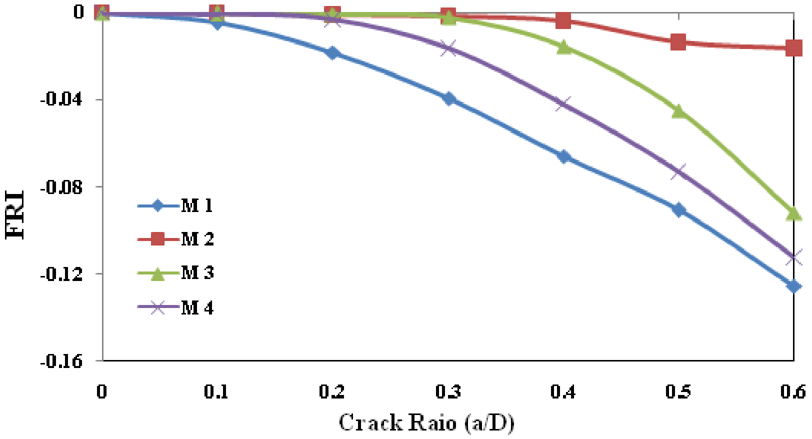

Figure 6: FRI against crack depth for first four modes

Tables 3 shows the magnitude of the reduction in stiffness as predicted with different crack depths based on the numerical results. The average of four modes where calculated and the result shows reduction in frequencies increased with increasing in the crack depths.

Table 3: The Reduction Magnitude in Natural Frequency with Increase in Crack Depth

Crack Depth (mm)	Mode 1	Mode 2	Mode 3	Mode 4	Reduction Magnitude
0	0	0	0	0	0
5	-0.457	-0.008	-0.036	-0.061	0.14%
10	-1.83	-0.114	-0.085	-0.305	0.58%
15	-3.93	-0.167	-0.221	-1.65	1.49%
20	-6.570	-0.388	-1.553	-4.21	3.18%
25	-9.055	-1.373	-4.5	-7.32	5.56%
30	-12.52	-1.642	-9.145	-11.23	8.63%

The reduction magnitude was average for all four modes and for each case. Figure 7 shows the relation between depth of cracks and reduction magnitude based on the numerical results. It is clear that, increasing depth of crack can lead to increase in the reduction magnitude.

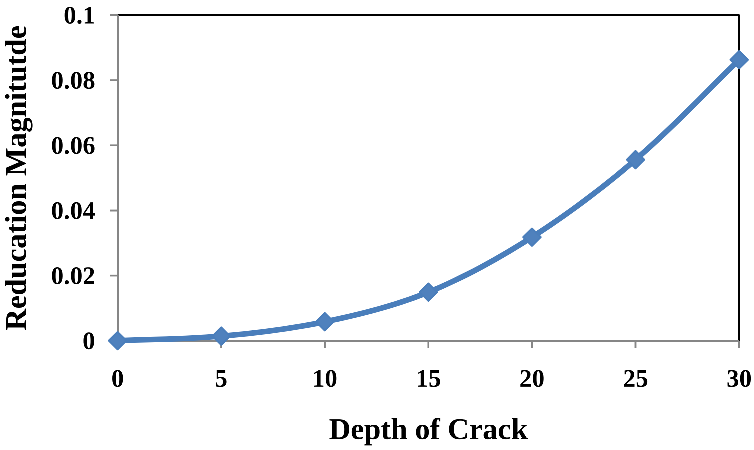

Figure 7: The relation between depth of cracks and the reduction magnitude.

6. Conclusion

Present study assessed the numerical analysis results based on Finite Element Modelling, by using theoretical model based on Euler–Bernoulli theory of the beam and equation of motion. The results show a good agreement between numerical and analytical results which lead to relay on the numerical analysis results.

The numerical analyses found that the depth of crack influenced the natural frequency, and reduced the magnitude of the natural frequency. The study showed small crack depth ratios had small effect on the sensitivity of the natural frequencies. It was also observed that the changes became more significant as the crack grew deeper. Based on this observation, it could be inadequate to use changes in natural frequencies alone in order to identify cracks in most real life situations.

Based on present study, it is suggested that Mode shape will be used for damage detection of small crack depth cases.

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