On Characterizations of Real Hypersurfaces in a Complex Space Form with n-Parallel Shape Operator

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On characterizations of real hypersurfaces in a complex space form with $\eta$-parallel shape operator

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Abstract

In this paper, we study real hypersurfaces in a non-flat complex space form with $\eta$-parallel shape operator. Several partial characterizations of these real hypersurfaces are obtained.

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1 Introduction

Let $M_n(c)$ be an $n$-dimensional complete and simply connected non-flat complex space form with constant holomorphic sectional curvature $4c$, i.e., it is either a complex projective space $\mathbb{C}P^n$ or a complex hyperbolic space $\mathbb{C}H^n$ (according to as the holomorphic sectional curvature $4c$ is positive or negative). Suppose $M$ is a connected real hypersurface in $M_n(c)$ and $N$ is a unit normal vector field of $M$. Then the complex structure $J$ of $M_n(c)$ induces an almost contact metric structure $(\phi, \xi, \eta, \langle, \rangle)$ on $M$, i.e.,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = \langle \xi, X \rangle.$$

We denote by $\Gamma(V)$ the module of all differentiable sections on the vector bundle $V$ over $M$. Typical examples of real hypersurfaces are the homogeneous real hypersurfaces $M$. In 1973, Takagi [17] classified these homogeneous real hypersurfaces in $\mathbb{C}P^n$ into six types, so-called real hypersurfaces of type $A_1$, $A_2$, $B$, $C$, $D$ and $E$. A Hopf hypersurface $M$ in $M_n(c)$ is characterized by the condition that the structure vector field $\xi$ is principal, i.e., $A\xi = \alpha \xi$, and it can be shown that this principal curvature $\alpha$ is a constant.

By looking at Takagi's classification, one may verify that the homogeneous real hypersurfaces are Hopf and with constant principal curvatures. In 1986, Kimura [7] showed that the converse is also true, i.e., Hopf hypersurfaces with constant principal
curvatures in $\mathbb{C}P^n$ are in fact those real hypersurfaces of type $A_1$, $A_2$, etc. Also, Berndt [2] showed a $\mathbb{C}H^n$ version for Kimura's result, i.e., Hopf hypersurfaces with constant principal curvatures could be divided into four types, nowadays known as type $A_0$, $A_1$, $A_2$ and $B$. In what follows, by real hypersurfaces of type $A$, we mean of type $A_1$, $A_2$ (resp. of type $A_0$, $A_1$, $A_2$) for $c > 0$ (resp. for $c < 0$). Other than these Hopf hypersurfaces, another example of real hypersurfaces in $M_n(c)$ are the class of ruled real hypersurfaces. Ruled real hypersurfaces in $M_n(c)$ are characterized by having a one-dimensional holonomy whose leaves are complex totally geodesic hyperplanes in $M_n(c)$. The geometry of ruled real hypersurfaces in $M_n(c)$ was studied in [10].

One of the first results in the theory of real hypersurfaces $M$ in $M_n(c)$ is the shape operator $A$ of $M$ in $M_n(c)$ cannot be parallel, i.e., $\nabla A \neq 0$, where $\nabla$ is the Levi-Civita connection of $M$. The non-existence of real hypersurfaces in $M_n(c)$ with parallel shape operator motivates the study of the weaker notion of $\eta$-parallelism, which was first introduced by Kimura and Maeda [8]. The shape operator $A$ is said to be $\eta$-parallel if it satisfies the following condition:

$$\langle (\nabla X) A, Z \rangle = 0$$

for any $X, Y$ and $Z \in \Gamma(D)$, where $D := \text{span}\{\xi \}$, called the holomorphic distribution on $M$. The complete classification of real hypersurfaces with $\eta$-parallel shape operator in $M_n(c)$ remain open up to this point, nevertheless, many partial characterizations have been obtained either by imposing an additional condition or by considering a condition that is slightly stronger than the $\eta$-parallelism (for instance, cf.[1, 4, 5, 8, 15, 16], etc). It is worthy to note that real hypersurfaces that appeared in the list of these characterizations are those of type $A$, $B$ and ruled real hypersurfaces.

In this paper, we shall continue the study of real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator. In particular, several partial characterizations of real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator are obtained.

This paper is organized as follows. Section 2 recalls some basic formulas and briefly reviews certain known results on real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator. Some auxiliary lemmas are derived in Section 3. In Section 4 we focus on contact real hypersurfaces in $M_n(c)$ and give a characterization for ruled real hypersurfaces and contact real hypersurfaces. In Section 5 we characterize real hypersurfaces in $M_n(c)$ with $\eta$-parallel shape operator under the commutativity assumption on $\phi A \phi$ and $\phi^2 A \phi^2$. In the last section we characterize real hypersurfaces in $M_n(c)$ with prescribed covariant derivative of the shape operator.

## 2 Preliminaries

Consider a connected real hypersurface $M$ in $M_n(c)$, the induced almost contact metric structure $(\phi, \xi, \eta, \langle , \rangle)$ on $M$ has the following properties

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1$$

$$\langle \nabla_X \phi \rangle Y = \eta(Y)AX - \langle AX, Y \rangle \xi, \quad \nabla_X \xi = \phi AX$$

$$\begin{align*}
\phi^2 X &= -X + \eta(X)\xi, \\
\phi \xi &= 0, \\
\eta(\phi X) &= 0, \\
\eta(\xi) &= 1
\end{align*}$$

$$\begin{align*}
\langle \nabla_X \phi \rangle Y &= \eta(Y)AX - \langle AX, Y \rangle \xi, \\
\nabla_X \xi &= \phi AX
\end{align*}$$
for any \(X, Y \in \Gamma(TM)\). Let \(R\) be the curvature tensor of \(M\). Then the equations of Gauss and Codazzi are given respectively by

\[
R(X, Y)Z = c(\langle Y, Z \rangle X - \langle X, Z \rangle Y + \langle \phi Y, Z \rangle \phi X - \langle \phi X, Z \rangle \phi Y
- 2\langle \phi X, Y \rangle \phi Z) + \langle AY, Z \rangle AX - \langle AX, Z \rangle AY
\]

\[
(\nabla_X A)Y - (\nabla_Y A)X = c(\eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y \rangle \xi).
\]

The second order covariant derivative \(\nabla_X \nabla_Y A\) on the shape operator \(A\) is defined by

\[
(\nabla_X \nabla_Y A)Z = \nabla_X((\nabla_Y A)Z) - (\nabla_{\nabla_X Y} A)Z - (\nabla_Y A)\nabla_X Z.
\]

Next, we state two necessary and sufficient conditions for real hypersurfaces in \(M_n(c)\) to be of type \(A\).

**Theorem 2.1 ([3, 11, 12, 14]).** Let \(M\) be a real hypersurface in \(M_n(c)\), \(n \geq 3\). Then the following are equivalent:

1. \(M\) is locally congruent to one of real hypersurfaces of type \(A\);
2. \(\phi A = A\phi\);
3. \((\nabla_X A)Y = -c(\langle \phi X, Y \rangle \xi + \eta(Y)\phi X)\), for any \(X, Y \in \Gamma(TM)\).

The following theorem, proved by Kimura and Maeda, and Suh respectively for \(c > 0\) and \(c < 0\), completely classified Hopf hypersurfaces with \(\eta\)-parallel shape operator in \(M_n(c)\).

**Theorem 2.2 ([8, 16]).** Let \(M\) be a Hopf hypersurface in \(M_n(c)\), \(n \geq 3\), with \(\eta\)-parallel shape operator. Then \(M\) is locally congruent to one of real hypersurfaces of type \(A\) and \(B\).

The above theorem is not true if the condition that \(M\) being Hopf is removed.

**Theorem 2.3 ([1, 8]).** Let \(M\) be a real hypersurface in \(M_n(c)\), \(n \geq 3\). Suppose \(M\) satisfies the following two conditions:

1. \(\phi(\phi A + A\phi)\phi = 0\), i.e., the holomorphic distribution \(D\) is integrable;
2. the shape operator \(A\) is \(\eta\)-parallel.

Then \(M\) is locally congruent to a ruled real hypersurface.

On the other hand, Ki and Suh studied real hypersurfaces \(M\) with \(\eta\)-parallel shape operator without assuming it is Hopf. By restricting Condition 2 and Condition 3 in Theorem 2.1 to the holomorphic distribution \(D\), they obtained the following result.

**Theorem 2.4 ([4]).** Let \(M\) be a real hypersurface in \(M_n(c)\), \(n \geq 3\). Suppose \(M\) satisfies the following two conditions:

1. \(\phi(\phi A - A\phi)\phi = 0\),
2. \((\nabla_X A)Y = -c(\phi X, Y)\xi\), for any \(X, Y \in \Gamma(D)\).

Then \(M\) is locally congruent to one of real hypersurfaces of type \(A\).
Observe that Condition 2 in this theorem is a special form for the shape operator $A$ to be $\eta$-parallel. Ahn, Lee and Suh weaken it to the $\eta$-parallelism condition on $A$ and proved the following.

**Theorem 2.5** ([1]). Let $M$ be a real hypersurface in $M_n(c), n \geq 3$. Suppose $M$ satisfies the following two conditions:

1. $\phi(\phi A - A\phi)\phi = 0$,

2. the shape operator $A$ is $\eta$-parallel.

Then $M$ is locally congruent to a ruled real hypersurface or one of real hypersurfaces of type $A$ and $B$.

The above theorem gave a significant improvement of Theorem 2.4 as it allows all the standard examples of real hypersurfaces with $\eta$-parallel shape operator to be included in the list of characterization.

Before we end this section, we shall state the expression of $\nabla A$ on these standard examples of real hypersurfaces with $\eta$-parallel shape operator.

**Theorem 2.6.** Let $M$ be real hypersurface in $M_n(c), n \geq 3$, and $X, Y \in \Gamma(D)$.

1. If $M$ is of type $A$ then 
   \[
   (\nabla_X A)Y = -c(\phi X, Y)\xi.
   \]

2. If $M$ is of type $B$ then 
   \[
   (\nabla_X A)Y = -c(\phi X, Y) + \frac{\alpha}{2}((\phi A - A\phi)X, Y)\xi.
   \]

3. If $M$ is ruled and $V = \phi A\xi$ then 
   \[
   (\nabla_X A)Y = -c(\phi X, Y) + \eta(AY)(X, V) + \eta(AX)(Y, V)\xi.
   \]

Statement 1 above is an immediate consequence of Statement 3 in Theorem 2.1 while Statement 3 above was derived in [15]. In order to verify Statement 2, we need to recall a lemma.

**Lemma 2.7** ([6]). Let $M$ be a real hypersurfaces in $M_n(c)$. If $A\xi = \alpha\xi$ then $\alpha$ is a constant and $(\nabla_\xi A) = (\alpha/2)(\phi A - A\phi)$.

Since the shape operator of real hypersurfaces of type $B$ is $\eta$-parallel, for $X, Y \in \Gamma(D)$ Statement 2 in the above theorem can be derived as follows:

\[
(\nabla_X A)Y = \langle(\nabla_X A)Y, \xi\rangle\xi \\
= \{-c(\phi X, Y) + (\nabla_\xi A)X, Y\}\xi \quad \text{(by the Codazzi equation)} \\
= \{-c(\phi X, Y) + \frac{\alpha}{2}((\phi A - A\phi)X, Y)\}\xi \quad \text{(by Lemma 2.7).}
\]
3 Real hypersurfaces with non-principal structure vector field

Hopf hypersurfaces with \( \eta \)-parallel shape operator \( A \) have already been completely characterized in Theorem 2.2. In this section, we focus on real hypersurfaces \( M \) on which the structure vector field \( \xi \) is not principal, or equivalently, with the restriction \( \beta := |\phi A \xi| \neq 0 \). Certain auxiliary lemmas that are needed in the following sections are also derived here.

We shall first fix some notations as follows: \( V := \nabla_\xi \xi = \phi A \xi, \alpha := \eta(A \xi) \) and \( F := \nabla_\xi A \). Then it is clear that the shape operator \( A \) of a real hypersurface \( M \) is \( \eta \)-parallel if and only if

\[
(\nabla_X A)Y = \{-c(\phi X,Y) + (FX,Y)\}\xi, \quad X,Y \in \Gamma(D).
\]

The next lemma plays an important role in the rest of the paper.

**Lemma 3.1.** Let \( M \) be a real hypersurface in \( M_n(c) \) with \( \eta \)-parallel shape operator \( A \). Then

\[
c\{ (Y, AZ)(X, W) - (X, AZ)(Y, W) \\
+ (\phi Y, AZ)(\phi X, W) - (\phi X, AZ)(\phi Y, W) - 2(\phi X, Y)(\phi AZ, W) \\
- (Y, Z)(X, AW) + (X, Z)(Y, AW) \\
- (\phi Y, Z)(\phi X, AW) + (\phi X, Z)(\phi Y, AW) + 2(\phi X, Y)(\phi Z, AW) \} \\
+ (AY, AZ)(AX, W) - (AX, AZ)(AY, W) \\
- (AY, Z)(AX, AW) + (AX, Z)(AY, AW) \\
= c\{ (Z, \phi AX)(\phi X, W) + (W, \phi AX)(\phi X, Z) \\
- (Z, \phi AX)(\phi Y, W) - (W, \phi AX)(\phi Y, Z) \} \\
+ (Y, \phi AX)(FZ, W) + (Z, \phi AX)(FY, W) + (W, \phi AX)(FZ, Y) \\
- (X, \phi AX)(FZ, W) - (Z, \phi AX)(FX, W) - (W, \phi AX)(FX, X)
\]

for any \( X, Y, Z, W \in \Gamma(D) \).

**Proof.** For any \( Y, Z, W \in \Gamma(D) \), by differentiating the following equation covariantly

\[
\langle (\nabla_Y A)Z, W \rangle = 0
\]

in the direction of \( X \in \Gamma(D) \), we obtain

\[
\langle (\nabla_X \nabla_Y A)Z + (\nabla_{\nabla_X Y} A)Z + (\nabla_Y A)\nabla_X Z, W \rangle + \langle (\nabla_Y A)Z, \nabla_X W \rangle = 0.
\]

From the \( \eta \)-parallelism condition and (2), the above equation reduces to

\[
\langle (\nabla_X \nabla_Y A)Z, W \rangle = (Y, \phi AX)\langle (\nabla_\xi A)Z, W \rangle + (Z, \phi AX)\langle (\nabla_\xi A)\xi, W \rangle \\
+ (W, \phi AX)\langle (\nabla_\xi A)\xi, Z \rangle.
\]
Furthermore, by using the Codazzi equation, the above equation becomes

\[
\langle (\nabla_X \nabla_Y A)Z, W \rangle = \langle Y, \phi AX \rangle \langle FZ, W \rangle + \langle Z, \phi AX \rangle \{ \langle FY, W \rangle - c\langle \phi Y, W \rangle \} \\
+ \langle W, \phi AX \rangle \{ \langle FY, Z \rangle - c\langle \phi Y, Z \rangle \}.
\]

Finally, by the Ricci identity \((R(X, Y)A)Z = (\nabla_X \nabla_Y A)Z - (\nabla_Y \nabla_X A)Z\) and the above equation, we obtain the lemma. □

**Lemma 3.2.** Let \( M \) be a real hypersurface in \( M_n(c) \) with \( \eta \)-parallel shape operator \( A \). Then

\[-(A\phi V, Y)(\phi V, X) + (A\phi V, X)(\phi V, Y) = \left( \frac{\tau}{2} (\phi A + A\phi)X + (F\phi A + A\phi F)X, Y \right)\]

for any \( X, Y \in \Gamma(D) \), where \( \tau := -\text{trace} \phi F \).

**Proof.** Let \( E_1, E_2, \ldots, E_{2n-2} \) be a local field of orthonormal frames in \( \Gamma(D) \). By putting \( Z = W = E_j \), for \( j = 1, 2, \ldots, 2n-2 \), in Lemma 3.1 and then summing up these equations, we get

\[
2(\phi A^2 Y, \phi AX) - 2(\phi A^2 X, \phi AY) = \sum_{j=1}^{2n-2} \langle FE_j, E_j \rangle (\phi AX + A\phi X, Y) \\
+ 2\langle FY, \phi AX \rangle - 2\langle FX, \phi AY \rangle.
\]

Next, by applying (1) in the left hand side of this equation, we obtain the lemma. □

**Lemma 3.3.** Let \( M \) be a real hypersurface in \( M_n(c) \), \( n \geq 3 \), with \( \eta \)-parallel shape operator \( A \). Suppose that \( \beta \) is nowhere zero on \( M \). If there exist two functions \( \nu \) and \( \bar{\nu} \) such that

\[AV = \nu V \quad \text{and} \quad A\phi V = \bar{\nu}\phi V - \beta^2 \xi\]

then \( \phi A\phi \) and \( \phi^2 A\phi^2 \) can be diagonalised simultaneously.

**Proof.** Let \( x \) be an arbitrary point in \( M \). From the hypothesis, the subspace

\[\mathcal{H} := \text{span}\{V, \phi V, \xi\}\]

and its orthogonal complement \( \mathcal{H}^\perp \) in \( T_xM \) are both invariant by \( A \) and hence by both \( \phi A\phi \) and \( \phi^2 A\phi^2 \) as well. Furthermore, each eigenvector \( E \in \mathcal{H}^\perp \) of \( \phi^2 A\phi^2 \) is a principal vector as well. If \( \phi E \) is principal, for each principal vector \( E \in \mathcal{H}^\perp \) then the statement is clearly true. Hence, we suppose that there is a unit principal vector \( E' \in \mathcal{H}^\perp \) but \( \phi E' \) is not principal.

Firstly, by letting \( X, Y, Z = V \) and \( W = \phi V \) in Lemma 3.1, we obtain

\[-2c\beta^2(\nu - \bar{\nu})(\phi X, Y) = (F\phi V, V)(\langle \phi A + A\phi \rangle X, Y). \tag{3}\]

Since \( \phi E' \) is not principal, we can see that \( \langle F\phi V, V \rangle = 0 = \nu - \bar{\nu} \). (for otherwise, by putting \( X = E' \) in the above equation, yields \( A\phi E' = \lambda \phi E' \) and a contradiction.)
Next, by putting $X = \phi V$, $Y = V$ in Lemma 3.1 and making use of the fact that $
u = \nu$, 

$$2c\beta^2\{(\phi A - A\phi)Z,W\} - \nu\beta^2\{(V,Z)\langle\phi V,W\rangle + \langle\phi V,Z\rangle\langle V,W\rangle\}$$

$$= -2\nu\beta^2\{FZ,W\} - \nu\{(V,Z)\langle FV,W\rangle + \langle V,W\rangle\langle FV,Z\rangle$$

$$+ \langle\phi V,Z\rangle\langle F\phi V,W\rangle + \langle\phi V,W\rangle\langle F\phi V,Z\rangle\}.$$  \hspace{1cm} (4)

If we put $Z,W \in \mathcal{H}^1$ in (4), then 

$$c((\phi A - A\phi)Z,W) = -\nu(FZ,W).$$  \hspace{1cm} (5)

From the hypothesis $\phi E'$ is not principal, the right hand side of (5) is not identically zero, so we may assume that $\nu \neq 0$. On the other hand, by putting $Z = V$ and $W = \phi V$ in (4), and taking account of $\langle FV,\phi V\rangle = \nu - \nu' = 0$, we obtain $-\nu\beta^2 = 0$. This contradicts the facts $\nu \neq 0$ and $\beta \neq 0$. The proof is completed. \hfill \Box

4 Characterizations on contact real hypersurfaces

An almost contact manifold $(M^{2n-1},\phi,\xi,\eta)$ is said to be a contact manifold if

$$\eta \wedge (d\eta)^{n-1} \neq 0$$

on $M$. If there is a Riemannian metric $\langle , \rangle$ which is compatible with this contact structure then $(\phi,\xi,\eta,\langle , \rangle)$ becomes a contact metric structure and $M$ is said to be a contact metric manifold.

A real hypersurface in a Kaehler manifold is said to be contact if its induced almost contact structure is contact. Okumura proved a necessary and sufficient condition for real hypersurfaces in a Kaehler manifold to be contact.

Theorem 4.1 ([13]). Let $M$ be a real hypersurface in a Kaehler manifold. Then the induced almost contact structure $(\phi,\xi,\eta)$ is contact if and only if there is a non-vanishing function $k$ on $M$ such that

$$\phi A + A\phi - k\phi = 0.$$  \hspace{1cm} (6)

It can be shown that $k$ is constant. Kon proved the following characterization while the ambient space is $CP^n$.

Theorem 4.2 ([9]). Let $M$ be a complete real hypersurface in $CP^n$, $n \geq 3$. If $M$ satisfies

$$\phi A + A\phi - \varepsilon\phi = 0$$

for some nonzero constant $\varepsilon$, then $M$ is congruent to one of real hypersurface of type $A_1$ and $B$.  

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On the other hand, Vernon gave a characterization of contact real hypersurfaces in $\mathbb{C}H^n$.

**Theorem 4.3** ([18]). *Let $M$ be a complete contact real hypersurface in $\mathbb{C}H^n$, $n \geq 3$. Then $M$ is congruent to one of real hypersurfaces of type $A_0$, $A_1$ and $B$.***

In this section, we study real hypersurfaces in $M_n(c)$ under a weaker version of (6), i.e.,

$$\phi(\phi A + A\phi - k\phi)\phi = 0,$$

(7)

for some function $k$ on $M$. We shall first derive some identities from the condition (7).

Note that (7) is equivalent to

$$\langle(\phi A + A\phi - k\phi)Y, Z\rangle = 0, \quad Y, Z \in \Gamma(D).$$

Differentiating this equation covariantly in the direction of $X \in \Gamma(D)$ we get

$$\langle\phi AY, \nabla_X Z\rangle + \langle(\nabla_X \phi)AY + \phi(\nabla_X A)Y + \phi A\nabla_X Y, Z\rangle$$

$$+ \langle A\phi Y, \nabla_X Z\rangle + \langle(\nabla_X A)\phi Y + A(\nabla_X \phi)Y + A\phi \nabla_X Y, Z\rangle$$

$$- \langle Xk(\phi Y, Z) - k(\phi Y, \nabla_X Z) - k(\nabla_X \phi)Y + \phi \nabla_X Y, Z\rangle = 0.$$

By using (2) and (7), this equation can be reformed as

$$-\langle Z, V\rangle \langle\phi AX, Y\rangle + \langle Y, V\rangle \langle\phi AX, Z\rangle - \langle(\nabla_X A)Y, \phi Z\rangle + \langle(\nabla_X A)Z, \phi Y\rangle$$

$$+ \eta(AY)\langle AX, Z\rangle - \eta(AZ)\langle AX, Y\rangle - \langle Xk(\phi Y, Z) = 0.$$}

Now by replacing $X, Y$ and $Z$ cyclically in the above equation and then summing these equations, with the help of the Codazzi equation and (7), we obtain

$$\mathcal{E}(k(X, V) + Xk)\langle\phi Y, Z\rangle = 0$$

where $\mathcal{E}$ denotes the cyclic sum over $X, Y$ and $Z$. Let $X$ be an arbitrary vector field in $\Gamma(D)$. If we choose $Y \perp X, \phi X$ and $Z = \phi Y$ in the above equation then $k(X, V) + Xk = 0$.

We summarize the above observations in the following lemma.

**Lemma 4.4.** *Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose $M$ satisfies

$$\phi(\phi A + A\phi - k\phi)\phi = 0$$

for some function $k$ on $M$. Then for any $X, Y$ and $Z \in \Gamma(D)$,

$$-\langle Z, V\rangle \langle\phi AX, Y\rangle + \langle Y, V\rangle \langle\phi AX, Z\rangle - \langle(\nabla_X A)Y, \phi Z\rangle + \langle(\nabla_X A)Z, \phi Y\rangle$$

$$+ \eta(AY)\langle AX, Z\rangle - \eta(AZ)\langle AX, Y\rangle - \langle Xk(\phi Y, Z) = 0,$$

(8)

$$k(X, V) + Xk = 0.$$*
We first look at the case where $k$ is a nonzero constant. In this case, the equation (9) implies that $V = 0$, i.e., $\xi$ is principal and so $(\phi A + A\phi - k\phi)\xi = 0$. Consequently, we have $(\phi A + A\phi - k\phi) = 0$, for some nonzero constant $k$, and hence it follows from Theorem 4.2 and Theorem 4.3 that we obtain

**Theorem 4.5.** Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$. If $M$ satisfies

$$\phi(\phi A + A\phi - e\phi)\phi = 0$$

for some constant $e \neq 0$, then $M$ is locally congruent to one of real hypersurface of type $A_0$, $A_1$ and $B$.

On the other hand, by adding the $\eta$-parallelism condition on the shape operator, we have the following characterization.

**Theorem 4.6.** Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$. Suppose $M$ satisfies the following two conditions:

(i) $\phi(\phi A + A\phi - k\phi)\phi = 0$, for some function $k$ on $M$;

(ii) the shape operator $A$ is $\eta$-parallel.

Then $M$ is locally congruent to a ruled real hypersurface or one of real hypersurface of type $A_0$, $A_1$ and $B$.

**Proof.** In this case, the equation (8) can be reduced as

$$-(Z,V)(\phi AX,Y) + (Y,V)(\phi AX,Z)$$

$$+ \eta(AY)(AX,Z) - \eta(AY)(AX,Y) - (Xk)(\phi Y,Z) = 0.$$ 

If we choose $Y \perp V$, $\phi V$ and $Z = \phi Y$ then $Xk = 0$, for all $X \in \Gamma(D)$ and together with (9), we obtain $k(V,V) = 0$ on $M$. Since we are studying local geometry, we may assume that either $k = 0$ on $M$ or $k$ is nowhere zero on $M$. If $k$ is identically zero then $M$ is ruled by Theorem 2.3. If $k$ is nowhere zero on $M$, $\xi$ is principal and so $(\phi A + A\phi - k\phi)\xi = 0$. Consequently, we have $(\phi A + A\phi - k\phi) = 0$, and hence our result follows from Theorem 4.2 and Theorem 4.3. 

\[ \square \]

5 Real hypersurfaces with a commutative condition

Observe that the Condition 1 in Theorem 2.3, Theorem 2.5 and Theorem 4.6 imply that $\phi^2 A\phi^2$ and $\phi A\phi$ are commutative. Hence, it is natural to ask if the Condition 1 in these theorems is replaceable by this condition. The main purpose of this section is to give an affirmative answer to this question. We first prove the following lemma.

**Lemma 5.1.** Let $M$ be a real hypersurface in $M_n(c)$, $n \geq 3$, with $\eta$-parallel shape operator $A$. If $\phi A\phi$ and $\phi^2 A\phi^2$ commute then either

(i) $\phi(\phi A - A\phi)\phi = 0$, or

(ii) $\phi(\phi A + A\phi - k\phi)\phi = 0$ for some function $k$ on $M$. 

\[ 9 \]
Proof. As $\phi A\phi$ and $\phi^2 A\phi^2$ are commutative, they can be diagonalized simultaneously and hence there is a local field of orthonormal frames $E_j, \phi E_j$ ($1 \leq j \leq n - 1$) on $\Gamma(D)$ such that

$$AE_j = e_j \xi + \lambda_j E_j$$
$$A\phi E_j = e_j \xi + \bar{\lambda}_j \phi E_j.$$  

By making the following substitutions for the vectors $X, Y, Z$ and $W$ in Lemma 3.1:

(a) $Y = Z = E_i, W = X = \phi E_j$, $i \neq j$;
(b) $Y = Z = E_i, W = X = E_j$, $i \neq j$;
(c) $Y = Z = E_i, W = X = \phi E_i$;
(d) $X = E_j, Y = \phi E_j, Z = \phi E_i, W = E_i$, $i \neq j$,

we obtain the following equations

$$\bar{\lambda}_j \lambda^2_j - (\lambda^2_j - c + e^2_j)\lambda_j + (e^2_j - c)\bar{\lambda}_j = 0 \quad (10)$$
$$\lambda_j \bar{\lambda}_j - (\lambda^2_j - c + e^2_j)\lambda_j + (e^2_j - c)\bar{\lambda}_j = 0 \quad (11)$$
$$(\lambda - \bar{\lambda}_i)(\lambda \bar{\lambda}_i + 5c) + \bar{\lambda}_i e^2_i - \lambda e^2_i + 2(E_{E_i}, \phi E_i)(\lambda_i + \bar{\lambda}_i) = 0 \quad (12)$$
$$2c(\lambda_i - \bar{\lambda}_i) + (\lambda_i + \bar{\lambda}_i)(E_{E_i}, \phi E_i) = 0. \quad (13)$$

If $\lambda_i = \bar{\lambda}_i$ for all $i$ then $\phi(\phi A - A\phi)\phi = 0$ and we obtain Statement (i). Hence, we suppose $\lambda_i \neq \bar{\lambda}_i$ for some $i$, says $\lambda_1 \neq \bar{\lambda}_1$. From (13), we obtain $(E_{E_i}, \phi E_i) \neq 0$ and

$$\lambda_r + \bar{\lambda}_r = 2c \frac{\bar{\lambda}_1 - \lambda_1}{(E_{E_i}, \phi E_i)}, \quad r \neq 1. \quad (14)$$

We consider two cases: (I) $\lambda_s \neq \bar{\lambda}_s$ for some $s \neq 1$; and (II) $\lambda_r = \bar{\lambda}_r$ for all $r \neq 1$.

Case (I) $\lambda_s \neq \bar{\lambda}_s$ for some $s \neq 1$, says $\lambda_2 \neq \lambda_2$.

From (13), we obtain $(E_{E_2}, \phi E_2) \neq 0$ and

$$\lambda_s + \bar{\lambda}_s = 2c \frac{\bar{\lambda}_2 - \lambda_2}{(E_{E_2}, \phi E_2)}, \quad s \neq 2. \quad (15)$$

By observing (14) and (15), we obtain

$$\lambda_i + \bar{\lambda}_i = 2c \frac{\bar{\lambda}_1 - \lambda_1}{(E_{E_1}, \phi E_1)}, \quad \text{for all } i. \quad (16)$$

Therefore, we obtain Statement (ii) with $k = 2c(\bar{\lambda}_1 - \lambda_1)(E_{E_1}, \phi E_1)^{-1}$.

Case (II) $\lambda_r = \bar{\lambda}_r$ for all $r \neq 1$.

In this case, (14) reduces to

$$\lambda_r = \bar{\lambda}_r = c \frac{\bar{\lambda}_1 - \lambda_1}{(E_{E_1}, \phi E_1)} \neq 0, \quad r \neq 1.$$
On the other hand, taking \( j = 1 \) and \( i \neq 1 \), and then by taking the operation \( \lambda_j \times (10) - \lambda_j \times (11) \), yields
\[
(\lambda_1 - \bar{\lambda}_1)(\lambda_1 \bar{\lambda}_1 + c) + \bar{\lambda}_1 \hat{e}_1^2 - \lambda_1 \hat{e}_1^2 = 0.
\]
From this equation and (12), we can see
\[
\lambda_1 + \bar{\lambda}_1 = 2c \frac{\bar{\lambda}_1 - \lambda_1}{(FE_1, \phi E_1)}.
\]
Adding this case into (14), we also obtain (16) and Statement (ii). This completes the proof.

It follows from Theorem 2.5, Theorem 4.6 and Lemma 5.1 that we have

**Theorem 5.2.** Let \( M \) be a real hypersurface in \( M_n(c) \), \( n \geq 3 \), with \( \eta \)-parallel shape operator \( A \). If \( \phi A \phi \) and \( \phi^2 A \phi^2 \) commute then \( M \) is locally congruent to a ruled real hypersurface or one of real hypersurface of type \( A \) and \( B \).

6 **Real hypersurfaces with prescribed covariant derivative of the shape operator**

In the previous sections, we characterized real hypersurfaces \( M \) with \( \eta \)-parallel shape operator \( A \) under certain additional conditions on \( M \). In this section we study these real hypersurfaces from another aspect, i.e., by looking at a condition that is slightly stronger than the \( \eta \)-parallelism on \( A \).

In Theorem 2.6 we see that these "standard examples" of real hypersurfaces with \( \eta \)-parallel shape operator have a nice form for the covariant derivative of the shape operator on the holomorphic distribution \( \mathcal{D} \). Motivated by these identities, it is natural to ask if the converse of the identities in Theorem 2.1 are true. In 1995, Suh proved the following

**Theorem 6.1 ([15]).** Let \( M \) be a real hypersurface in \( M_n(c) \), \( n \geq 3 \). If \( M \) satisfies
\[
(\nabla_X A)Y = \{-c(\phi X, Y) + \eta(AY)(X, V) + \eta(AX)(Y, V)\} \xi
\]
for any \( X, Y \in \Gamma(\mathcal{D}) \), then \( M \) is locally congruent to a ruled real hypersurface or a real hypersurface of type \( A \).

It follows from the above theorem that, since \( V = 0 \) is necessary and sufficient for \( \xi \) to be principal, we can easily obtain the following characterization for real hypersurfaces of type \( A \).

**Corollary 6.2.** Let \( M \) be a real hypersurface in \( M_n(c) \), \( n \geq 3 \). Suppose \( M \) satisfies
\[
(\nabla_X A)Y = -c(\phi X, Y) \xi
\]
for any \( X, Y \in \Gamma(\mathcal{D}) \). Then \( M \) is locally congruent to a real hypersurface of type \( A \).
The condition in Theorem 6.1 is too strong to be used to characterize all the standard examples of real hypersurfaces with \( \eta \)-parallel shape operator. It shall be replaced by a weaker condition in order to broaden the list of characterization. In this sense, we have the following.

**Theorem 6.3.** Let \( M \) be a real hypersurface in \( M_n(c) \), \( n \geq 3 \). Suppose \( M \) satisfies

\[
(\nabla_X A)Y = \{-c(\phi X, Y) + \eta(AY)(X, V) + \eta(AX)(Y, V) \\
+ \epsilon((\phi A - A\phi)X, Y)\} \xi
\]

(17)

for any \( X, Y \in \Gamma(D) \), where \( \epsilon \) is a constant. Then \( M \) is locally congruent to a ruled real hypersurface or one of real hypersurfaces of type \( A \) and \( B \).

**Proof.** The condition (17) implies that \( A \) is \( \eta \)-parallel. If \( \xi \) is principal then by virtue of Theorem 2.1 and Theorem 2.2, we conclude that \( M \) is of type \( A \) or \( B \). Hence, we may suppose that \( \beta \) is nowhere zero on \( M \). On the other hand, with the condition (17), the tensor field \( F \) takes the form

\[
\langle FX, Y \rangle = \eta(AY)(X, V) + \eta(AX)(Y, V) + \epsilon((\phi A - A\phi)X, Y)
\]

(18)

for any \( X, Y \in \Gamma(D) \). It follows from this equation that \( \tau = -\text{trace}\,\phi F\phi = 0 \). Moreover, the identity in Lemma 3.2 can be reduced to

\[
-\langle AX, V \rangle(Y, V) + \langle AY, V \rangle(X, V) = \epsilon((A\phi A\phi - \phi A\phi A)X, Y).
\]

(19)

First, by putting \( X = V \) and \( Y = \phi V \) in (19), we obtain \( \langle AV, \phi V \rangle = 0 \). Next, if we put \( Y = \phi V \) in (19) then

\[
\epsilon((A\phi A + \phi A\phi A)\phi V, X) = 0
\]

(20)

for any \( X \in \Gamma(D) \). Finally, when we put \( Y = V \) in (19), we get

\[
\beta^2\langle AX, V \rangle - \langle AV, V \rangle(X, V) = \epsilon(\phi X, (A\phi A + \phi A\phi A)\phi V)
\]

\[
= 0 \quad \text{(from (20)).}
\]

This equation tells us that \( AV = \nu V \). Next, we wish to prove that \( A\phi V = \nu \phi V - \beta^2 \xi \).

For this purpose, we put \( Y = \phi V \) and \( Z = W = V \) in Lemma 3.1, then

\[
0 = c\{\beta^2(A\phi V, X) - \langle A\phi V, \phi V \rangle(\phi V, X)\} + \frac{\langle FV, V \rangle}{2}(\phi A\phi V - \nu V, X)
\]

\[
- \langle A\phi V, \phi V \rangle(FV, X) + \langle FV, \phi V \rangle(A\phi V, X).
\]

On the other hand, by putting \( Y = V \) and \( Z = W = \phi V \) in Lemma 3.1, we get

\[
c\{\beta^2(A\phi V, \phi V) - \langle A\phi V, \phi V \rangle(V, X)\} = \frac{\langle FV, \phi V \rangle}{2}(\nu \phi V + A\phi V, X)
\]

\[
+ \nu\{\beta^2(F\phi V, X) - \langle FV, \phi V \rangle(V, X)\}.
\]

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By using (18), the above two equations becomes

\[
\begin{align*}
(\beta^2 - \epsilon \nu - c)\{\beta^2 (A\phi V, X) - (A\phi V, \phi V)(\phi V, X)\} &= 0 \\
(\epsilon \nu + c)\{\beta^2 (A\phi V, \phi X) - (A\phi V, \phi V)(\phi V, \phi X)\} &= 0
\end{align*}
\]

for any \(X \in \Gamma(D)\). From these two equations and the fact that \(\beta \neq 0\),

\[
(A\phi V, X) = \beta^{-2} (A\phi V, \phi V)(\phi V, X), \quad X \in \Gamma(D)
\]

and hence we have \(A\phi V = \tilde{\nu}\phi V - \beta^2 \xi\), where \(\tilde{\nu} = \beta^{-2} (A\phi V, \phi V)\). According to Lemma 3.3 and Theorem 5.2, we conclude that \(M\) is ruled and this completes the proof. \(\square\)

References


