Simultaneous Linear Circular Functional Relationship Model

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Abstract
This paper proposed the statistical model on how to look at the relationship between several circular variables which area subject to measurements errors. Maximum likelihood estimation of parameters has been obtained iteratively assuming the ratio of concentration parameters are known and by choosing suitable initial values. The variance and covariance of parameters have also been derived using the Fisher information matrix. The model was applied to the Malaysian wind direction data recorded at various levels.

Introduction

Circular variables can be defined as one which takes values on the circumference of a circle and they are in the form of angles in the range $(0^0, 360^0)$ or $(0, 2\pi)$ radians. This type of data are widely use in geological, meteorological, biological and astronomical. A simple linear circular functional relationship model refer to a two mathematical circular variables X and Y which are observed inexactly, linearly related by $Y = \alpha + \beta X$. It also assume that the errors of circular variables X and Y are independently distributed and follow the von Mises with probability density function given by Mardia (1972). It can be defined as $g(\mu_0, \kappa; \theta) = (2\pi I_0(\kappa))^{-1} \exp\{\kappa \cos(\theta - \mu_0)\}$ where $I_0(\kappa)$ is the modified Bessel function of the first kind and order zero. The parameter μ_0 is the mean direction while the parameter κ is described as the concentration parameter. In this paper we consider a linear functional relationship between all pairs of a set of $(q > 2)$ random variables $X_i (i = 0, \ldots, q-1)$. This is also known as the simultaneous linear circular functional relationship model and can be used, as an example in assessing the relative relationship of a set of q random variables. As an illustration, this model will be used in studying the fundamental relationship of Malaysian wind direction data which have been recorded at different levels and locations.

The model

Suppose the variables $Y_j (j = 1, \ldots, q)$ are related to X by the linear circular functional relationship $Y_j = \alpha_j + \beta_j X$. Let (X_i, Y_{ji}) be the true values of the circular variables X and Y_j respectively and we assume that the observations x_i and y_{ji} have been measured with errors δ_i and ε_{ji} respectively. Thus, the full model can be written as

$$x_i = X_i + \delta_i \text{ and } y_{ji} = Y_{ji} + \varepsilon_{ji}, \text{ where}$$

$$Y_j = \alpha_j + \beta_j X (\text{mod } 2\pi), \text{ for } j = 1, \ldots, q$$

We also assume δ_i and ε_{ij} are independently distributed with von Mises distribution that is,

$$\delta_i \sim VM(0, \kappa) \text{ and } \varepsilon_{ji} \sim VM(0, \nu_j).$$

Suppose that we assume the ratio of the error concentration parameters for circular functional relationship model that $\lambda_j = \dfrac{\nu_j}{\kappa}$ is known. Then, the likelihood function is,

$$L(\alpha_j, \beta_j, \kappa, X_1, \ldots, X_n; \lambda_j, x_1, \ldots, x_n, y_{11}, \ldots, y_{qn})$$

$$= (2\pi)^{-2n} I_0^{-n}(\kappa) \sum_{j=1}^{q} I_0^{-n}(\lambda_j \kappa) \sum_{i=1}^{n} \exp\{\kappa \cos(\delta_i)\} \sum_{j=1}^{q} \exp\left\{\lambda_j \kappa \sum_{i=1}^{n} \cos(\varepsilon_{ji})\right\}$$

$$= (2\pi)^{-2n} I_0^{-n}(\kappa) \sum_{j=1}^{q} I_0^{-n}(\lambda_j \kappa) \sum_{i=1}^{n} \exp\{\kappa \cos(x_i - X_i)\} \sum_{j=1}^{q} \exp\left\{\lambda_j \kappa \sum_{i=1}^{n} \cos(y_{ji} - \alpha_j - \beta_j X_i)\right\}$$

By taking log to likelihood function, the equation above becomes,

$$\log L(\alpha_j, \beta_j, \kappa, X_1, \ldots, X_n; \lambda_j, x_1, \ldots, x_n, y_{11}, \ldots, y_{qn}) =$$

$$-2n\log(2\pi) - n\log I_0(\kappa) - n\sum_{j=1}^{q} \log I_0(\lambda_j \kappa) + \kappa \sum_{i=1}^{n} \cos(x_i - X_i) + \kappa \sum_{j=1}^{q} \lambda_j \sum_{i=1}^{n} \cos(y_{ji} - \alpha_j - \beta_j X_i)$$

By assuming that $\lambda_j = 1$, the equation above can be simplified as below,

$$-2n\log(2\pi) - (1+q)n\log I_0(\kappa) + \kappa \sum_{i=1}^{n} \cos(x_i - X_i) + \kappa \sum_{j=1}^{q} \sum_{i=1}^{n} \cos(y_{ji} - \alpha_j - \beta_j X_i)$$

Parameter estimation

There are $(2q+n+1)$ parameters to be estimated, i.e. $\alpha_1, \ldots, \alpha_q$, $\beta_1, \ldots, \beta_q$, κ and incidental parameters $X_1, \ldots, X_n$ by using maximum likelihood estimation (MLE). By differentiating $\log L$ with respect to the parameters α_j, β_j, κ and X_i, we can obtain the parameters $\hat{\alpha}_j$, $\hat{\beta}_j$, $\hat{\kappa}$ and $\hat{X}_i$ as the followings :

(i) MLE for $\hat{\alpha}_j$

The first partial derivative of the log likelihood function with respect to α_j is

$$\frac{\partial \log L}{\partial \alpha_j} = \kappa \sum_i \sin(y_{ji} - \alpha_j - \beta_j X_i)$$

By setting $\frac{\partial \log L}{\partial \alpha_j} = 0$ and simplifying the equation we get,

$$\tan \hat{\alpha}_j = \frac{\sum_i \sin(y_{ji} - \hat{\beta}_j \hat{X}_i)}{\sum_i \cos(y_{ji} - \hat{\beta}_j \hat{X}_i)}$$

$$\hat{\alpha}_j = \tan^{-1}\left\{\frac{\sum_i \sin(y_{ji} - \hat{\beta}_j \hat{X}_i)}{\sum_i \cos(y_{ji} - \hat{\beta}_j \hat{X}_i)}\right\}$$

$$= \tan^{-1}\left\{\frac{S}{C}\right\} \quad \text{where } S = \sum_i \sin(y_{ji} - \hat{\beta}_j \hat{X}_i) \text{ and } C = \sum_i \cos(y_{ji} - \hat{\beta}_j \hat{X}_i).$$

That is,

$$\hat{\alpha}_j = \begin{cases} \tan^{-1}\left\{\frac{S}{C}\right\} & S > 0, C > 0 \\ \tan^{-1}\left\{\frac{S}{C}\right\} + \pi & C < 0 \\ \tan^{-1}\left\{\frac{S}{C}\right\} + 2\pi & S < 0, C > 0 \end{cases}$$

(ii) MLE for $\hat{\beta}_j$

The first partial derivative of the log likelihood function with respect to β_j is,

$$\frac{\partial \log L}{\partial \beta_j} = \kappa \sum_i X_i \sin\left(y_{ji} - \alpha_j - \beta_j X_i\right)$$

$\hat{\beta}_j$ cannot be obtained analytically and may be obtained iteratively. By setting $\frac{\partial \log L}{\partial \beta_j} = 0$ and suppose $\hat{\beta}_{j0}$ is an initial estimate of $\hat{\beta}_j$. Then we can simplify the equation and approximately given by:

$$\hat{\beta}_{j1} \approx \hat{\beta}_{j0} + \frac{\sum_i \hat{X}_i \sin\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_{j0}\hat{X}_i\right)}{\sum_i \hat{X}_i^2 \cos\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_{j0}\hat{X}_i\right)}$$

where $\hat{\beta}_{j1}$ is an improvement of $\hat{\beta}_{j0}$.

(iii) MLE for $\hat{X}_i$

The first partial derivative of the log likelihood function with respect to X_i is,

$$\frac{\partial \log L}{\partial X_i} = \kappa \sin(x_i - X_i) + \kappa \sum_j \beta_j \sin\left(y_{ji} - \alpha_j - \beta_j X_i\right).$$

As for $\hat{\beta}_j$, the parameter $\hat{X}_i$ may also obtained iteratively. By setting $\frac{\partial \log L}{\partial X_i} = 0$ and suppose X_{i0} is an initial estimate of $\hat{X}_i$, we get the equation approximately given by:

$$\frac{\sin\left(x_i - \hat{X}_{i0}\right) + \sum_j \hat{\beta}_j \sin\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_{i0}\right)}{\cos\left(x_i - \hat{X}_{i0}\right) + \sum_j \hat{\beta}_j^2 \cos\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_{i0}\right)} + \Delta_i = 0 \qquad \text{where } \Delta_i = \hat{X}_{i0} - \hat{X}_{i1}$$

$$\frac{\sin\left(x_i - \hat{X}_{i0}\right) + \sum_j \hat{\beta}_j \sin\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_{i0}\right)}{\cos\left(x_i - \hat{X}_{i0}\right) + \sum_j \hat{\beta}_j^2 \cos\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_{i0}\right)} + \left(\hat{X}_{i0} - \hat{X}_{i1}\right) = 0$$

$$\hat{X}_{i1} \approx \hat{X}_{i0} + \frac{\sin\left(x_i - \hat{X}_{i0}\right) + \sum_j \hat{\beta}_j \sin\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_{i0}\right)}{\cos\left(x_i - \hat{X}_{i0}\right) + \sum_j \hat{\beta}_j^2 \cos\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_{i0}\right)}$$

where $\hat{X}_{i1}$ is an improvement of $\hat{X}_{i0}$.

(iv) MLE for $\hat{\kappa}$

The first partial derivative of the log likelihood function with respect to κ is,

$$\frac{\partial \log L}{\partial \kappa} = -n(1+q)\frac{I_0'(\kappa)}{I_0(\kappa)} + \sum_i \cos(x_i - X_i) + \sum_j \sum_i \cos\left(y_{ji} - \alpha_j - \beta_j X_i\right)$$

By setting $\frac{\partial \log L}{\partial \kappa} = 0$ and simplify to get the equation approximately given by,

$$A(\hat{\kappa}) = \frac{1}{n(1+q)}\left\{\sum_i \cos\left(x_i - \hat{X}_i\right) + \sum_j \sum_i \cos\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_i\right)\right\}$$

where $A(\hat{\kappa}) = \frac{I_0'(\hat{\kappa})}{I_0(\hat{\kappa})} = \frac{I_1(\hat{\kappa})}{I_0(\hat{\kappa})}$

Hence,

$$\hat{\kappa} = A^{-1}\left[\frac{1}{n(1+q)}\left\{\sum_i \cos\left(x_i - \hat{X}_i\right) + \sum_j \sum_i \cos\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_i\right)\right\}\right].$$

Estimation of $\hat{\kappa}$ can be obtained by using the approximation given by Fisher,

$$A^{-1}(w) = \begin{cases} 2w + w^3 + \frac{5}{6}w^5 & w < 0.53 \\ -0.4 + 1.39w + \frac{0.43}{(1-w)} & 0.53 \le w < 0.85 \\ \frac{1}{w^3 - 4w^2 + 3w} & w \ge 0.85 \end{cases}$$

Thus,

$$\hat{\kappa} = A^{-1}(w) \text{ where } w = \frac{1}{n(1+q)}\left\{\sum_i \cos\left(x_i - \hat{X}_i\right) + \sum_j \sum_i \cos\left(y_{ji} - \hat{\alpha}_j - \hat{\beta}_j \hat{X}_i\right)\right\}.$$

Fisher information matrix of parameters

We will consider the Fisher information matrix of $\hat{\kappa}$, $\hat{\alpha}_j$ and $\hat{\beta}_j$ for the simultaneous circular functional relationship model assuming the ratio of the error concentration parameters, λ_j is known. By considering the first partial derivatives of the log likelihood function as well as the second derivatives and their negative expected values, we define the Fisher information matrix, F for $\hat{\kappa}, \hat{\alpha}_j$ and $\hat{\beta}_j$ which are given by

$$F = \begin{bmatrix} R & 0 & W \\ 0 & S & 0 \\ W^T & 0 & U \end{bmatrix}$$

where R is an $n \times n$ matrix given by

$$R=\begin{bmatrix}\hat{\kappa}A(\hat{\kappa})+\hat{\kappa}\sum_{j}\lambda_j\hat{\beta}_j^2A(\lambda_j\hat{\kappa}) & & 0\\ & \ddots & \\ 0 & & \hat{\kappa}A(\hat{\kappa})+\hat{\kappa}\sum_{j}\lambda_j\hat{\beta}_j^2A(\lambda_j\hat{\kappa})\end{bmatrix}$$

W is an $n\times 2$ matrix given by

$$W=\begin{bmatrix}\hat{\kappa}\lambda_j\hat{\beta}_jA(\lambda_j\hat{\kappa}) & \hat{\kappa}\lambda_j\hat{\beta}_j\hat{X}_1A(\lambda_j\hat{\kappa})\\ \vdots & \vdots\\ \hat{\kappa}\lambda_j\hat{\beta}_jA(\lambda_j\hat{\kappa}) & \hat{\kappa}\lambda_j\hat{\beta}_j\hat{X}_nA(\lambda_j\hat{\kappa})\end{bmatrix}$$

S is given by $S=n\left(A'(\hat{\kappa})+\sum_{j}\lambda_j^2A'(\lambda_j\hat{\kappa})\right)$ where $A'(\kappa)=1-A^2(\kappa)-\frac{A(\kappa)}{\kappa}$ and

U is 2×2 matrix given by

$$U=\begin{bmatrix}n\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa}) & \lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\sum_{i}\hat{X}_i\\ \lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\sum_{i}\hat{X}_i & \lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\sum_{i}\hat{X}_i^2\end{bmatrix}.$$

The bottom right minor of the inverse of F with order 3×3 will form the asymptotic covariance matrix of $\hat{\kappa}$, $\hat{\alpha}_j$ and $\hat{\beta}_j$. From the theory of partitioned matrices, this is given by

$$V\hat{a}r\begin{bmatrix}\hat{\kappa}\\ \hat{\alpha}_j\\ \hat{\beta}_j\end{bmatrix}=\begin{bmatrix}S^{-1} & 0\\ 0 & \left(U-W^TR^{-1}W\right)^{-1}\end{bmatrix}$$

where $S^{-1}=\dfrac{1}{n\left(A'(\hat{\kappa})+\sum_{j}\lambda_j^2A'(\lambda_j\hat{\kappa})\right)}$ and $A'(\kappa)=1-A^2(\kappa)-\frac{A(\kappa)}{\kappa}$

Furthermore,

$$S^{-1}=\frac{1}{n\left[1-A^2(\hat{\kappa})-\frac{A(\hat{\kappa})}{\hat{\kappa}}+\sum_{j}\lambda_j^2\left(1-A^2(\lambda_j\hat{\kappa})-\frac{A(\lambda_j\hat{\kappa})}{\lambda_j\hat{\kappa}}\right)\right]}$$

$$=\frac{\hat{\kappa}}{n\left[\hat{\kappa}-\hat{\kappa}A^2(\hat{\kappa})-A(\hat{\kappa})+\sum_{j}\lambda_j\left(\lambda_j\hat{\kappa}-\lambda_j\hat{\kappa}A^2(\lambda_j\hat{\kappa})-A(\lambda_j\hat{\kappa})\right)\right]}$$

Thus,

$$(U-W^TR^{-1}W)^{-1}=\frac{M^2}{P^2\left(M-\hat{\beta}_j^2P\right)^2\left\{n\sum_i\hat{X}_i^2-\left(\sum_i\hat{X}_i\right)^2\right\}}\left\{\frac{P\left(M-\hat{\beta}_j^2P\right)}{M}\begin{bmatrix}\sum_i\hat{X}_i^2 & -\sum_i\hat{X}_i\\ -\sum_i\hat{X}_i & n\end{bmatrix}\right\}$$

$$=\frac{M}{P\left(M-\hat{\beta}_j^2P\right)\left\{n\sum_i\hat{X}_i^2-\left(\sum_i\hat{X}_i\right)^2\right\}}\begin{bmatrix}\sum_i\hat{X}_i^2 & -\sum_i\hat{X}_i\\ -\sum_i\hat{X}_i & n\end{bmatrix}$$

$$=Q\begin{bmatrix}\sum_i\hat{X}_i^2 & -\sum_i\hat{X}_i\\ -\sum_i\hat{X}_i & n\end{bmatrix}$$

where,

$P=\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})$ and

$M=\hat{\kappa}A(\hat{\kappa})+\sum_j\hat{\beta}_j^2P$

$$Q=\frac{\hat{\kappa}A(\hat{\kappa})+\sum_j\beta_j^2\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})}{\left[\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\right]\left(\hat{\kappa}A(\hat{\kappa})+\sum_j\hat{\beta}_j^2\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})-\hat{\beta}_j^2\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\right)\left\{n\sum_i\hat{X}_i^2-\left(\sum_i\hat{X}_i\right)^2\right\}}$$

Therefore, the asymptotic covariance matrix for $\hat{\kappa}$, $\hat{\alpha}_j$ and $\hat{\beta}_j$ are given by,

$$G=\begin{bmatrix}\left\{n\left[A'(\hat{\kappa})+\sum_j\lambda_j^2A'(\lambda_j\hat{\kappa})\right]\right\}^{-1} & 0 & 0\\ 0 & Q\sum_i\hat{X}_i^2 & -Q\sum_i\hat{X}_i\\ 0 & -Q\sum_i\hat{X}_i & Qn\end{bmatrix}$$

In particular we have the following results,

$$\hat{v}ar(\hat{\kappa})=\frac{\hat{\kappa}}{n\left\{\hat{\kappa}-\hat{\kappa}A^2(\hat{\kappa})-A(\hat{\kappa})+\sum_j\lambda_j\left[\lambda_j\hat{\kappa}-\lambda_j\hat{\kappa}A^2(\lambda_j\hat{\kappa})-A(\lambda_j\hat{\kappa})\right]\right\}}$$

$$\hat{v}ar(\hat{\alpha}_j)=\frac{\left(\hat{\kappa}A(\hat{\kappa})+\sum_j\hat{\beta}_j^2\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\right)\sum_i\hat{X}_i^2}{\left[\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\right]\left(\hat{\kappa}A(\hat{\kappa})+\sum_j\hat{\beta}_j^2\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})-\hat{\beta}_j^2\lambda_j\hat{\kappa}A(\lambda_j\hat{\kappa})\right)\left\{n\sum_i\hat{X}_i^2-\left(\sum_i\hat{X}_i\right)^2\right\}}$$

$$V\hat{a}r\left(\hat{\beta}_j\right)=\frac{\left(\hat{\kappa}A\left(\hat{\kappa}\right)+\sum_j \hat{\beta}_j^2\lambda_j\hat{\kappa}A\left(\lambda_j\hat{\kappa}\right)\right)n}{\left[\lambda_j\hat{\kappa}A\left(\lambda_j\hat{\kappa}\right)\right]\left(\hat{\kappa}A\left(\hat{\kappa}\right)+\sum_j \hat{\beta}_j^2\lambda_j\hat{\kappa}A\left(\lambda_j\hat{\kappa}\right)-\hat{\beta}_j^2\lambda_j\hat{\kappa}A\left(\lambda_j\hat{\kappa}\right)\right)\left\{n\sum_i \hat{X}_i^2-\left(\sum_i \hat{X}_i\right)^2\right\}}$$

By assuming $\lambda_j = 1$, the above equation may be simplify to get,

$$V\hat{a}r\left(\hat{\kappa}\right)=\frac{\hat{\kappa}}{n\left(1+q\right)\left[\hat{\kappa}-\hat{\kappa}A^2\left(\hat{\kappa}\right)-A\left(\hat{\kappa}\right)\right]}$$

$$V\hat{a}r\left(\hat{\alpha}_j\right)=\frac{\left(1+\sum_j \hat{\beta}_j^2\right)\sum_i \hat{X}_i^2}{\hat{\kappa}A\left(\hat{\kappa}\right)\left(1+\sum_j \hat{\beta}_j^2-\hat{\beta}_j^2\right)\left\{n\sum_i \hat{X}_i^2-\left(\sum_i \hat{X}_i\right)^2\right\}}$$

$$V\hat{a}r\left(\hat{\beta}_j\right)=\frac{\left(1+\sum_j \hat{\beta}_j^2\right)n}{\hat{\kappa}A\left(\hat{\kappa}\right)\left(1+\sum_j \hat{\beta}_j^2-\hat{\beta}_j^2\right)\left\{n\sum_i \hat{X}_i^2-\left(\sum_i \hat{X}_i\right)^2\right\}}$$

Numerical examples

The wind direction data was collected at two different locations and three different levels. The first data set was collected daily on March and April 2002 from TM Telecommunication Tower at Seberang Jaya, Penang. A total of 24 observations have been recorded at three different levels with their corresponding height as follows:

Variables	Level & Height
X	A & 45.72 m
Y1	B & 75.28 m
Y2	C & 97.28 m

The second dataset was recorded at Bayan Lepas airport on July and August 2005 at time 1200, located at 16.3 m above ground level, latitude 05°18'N and longitude 100°16'E. A total of 62 observations have been recorded at three different pressures with their corresponding height as follows:

Variables	Preasure & Height
X	850 Hpa & 5000 m
Y1	1000 Hpa & 300 m
Y2	500 Hpa & 19000 m

From the spoke plot below we may suggest that there is a linear relationship between those variables involved in the measurement of data set.

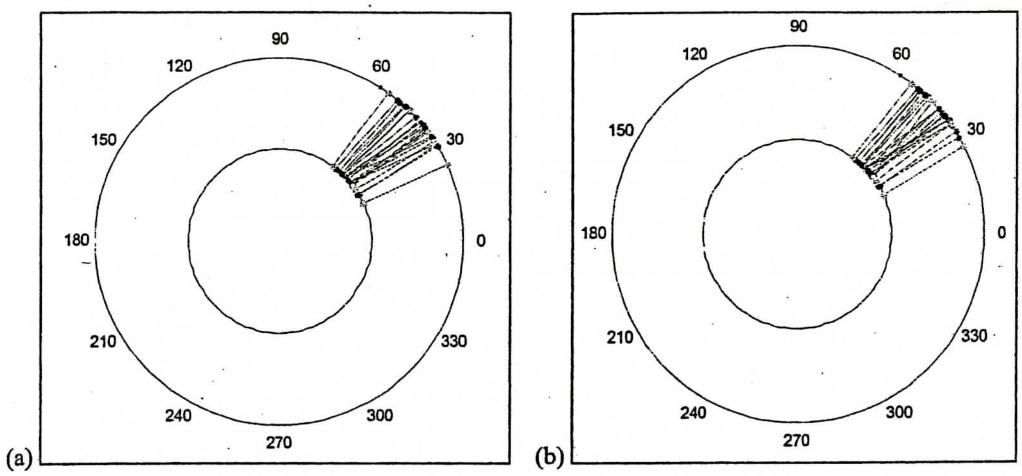

Figure 1: Spoke plot for Seberang Jaya (a) X & Y1 (b) X & Y2

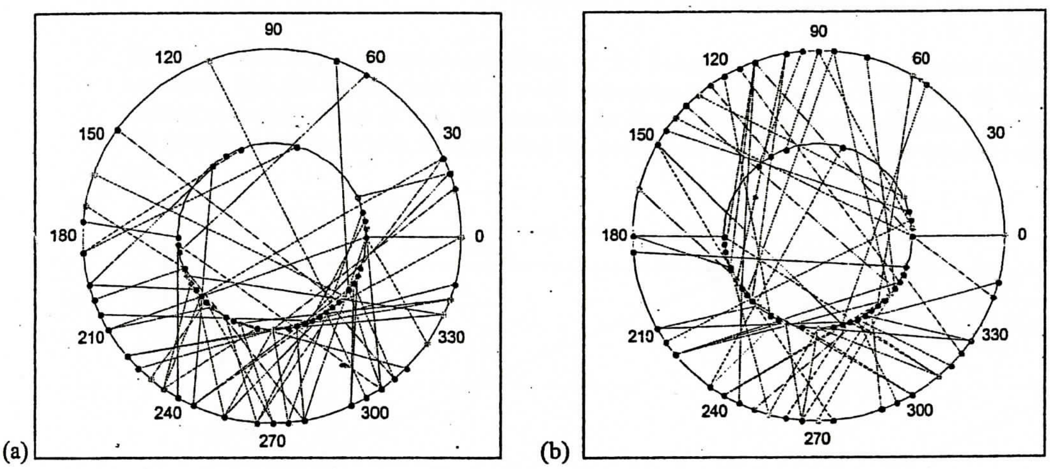

Figure 2: Spoke plot for Bayan Lepas (a) X & Y1 (b) X & Y2

In these examples the maximum likelihood estimates have been obtained by assuming equal error of the concentration parameters. For the convergence criteria, we set that the iteration will stop once our β_j change by no more than 0.0001. We start the iteration by selecting $\hat{X}_{i0} = x_i$ and $\hat{\beta}_0 = 1$ as initial values. These values are chosen because we expect that $\hat{X}_i$ should be close to $x_i, i = 1,\ldots,n$ and $\hat{\beta}_j$ around 1. Our aim is to find α_1, α_2, β_1, β_2 and κ which will maximize the log likelihood function as well as getting the standard error for each parameter estimates by using the Fisher information as described previously. Table below shown the maximum likelihood estimates and their standard error for each data set.

Table 1: Parameter estimation for Seberang Jaya and Bayan Lepas data set.

Parameter	Estimate (Standard error)	
	Seberang Jaya	Bayan Lepas
$\hat{\alpha}_1$	6.2275 (5.8345x10^{-2})	6.0724 (5.3919x10^{-1})
$\hat{\alpha}_2$	0.0335 (4.8675x10^{-2})	6.1092 (4.4381x10^{-1})
$\hat{\beta}_1$	1.0263 (7.9180x10^{-2})	1.0340 (1.2032x10^{-1})
$\hat{\beta}_2$	0.9387 (6.6057x10^{-2})	0.9119 (9.9037x10^{-2})
$\hat{\kappa}$	991.4442 (1.6520x10^{2})	1.5389 (1.5070x10^{-1})

In particular, the relationship for wind direction data at Seberang Jaya is given by

$$Y_1 = 6.228 + 1.026X \pmod{2\pi}$$
$$Y_2 = 0.034 + 0.939X \pmod{2\pi}$$

while the relationship for wind direction at Bayan Lepas is given by

$$Y_1 = 6.072 + 1.034X \pmod{2\pi}$$
$$Y_2 = 6.109 + 0.912X \pmod{2\pi}$$

The large value of parameter concentration, κ for Seberang Jaya data set compared to Bayan Lepas data set shows that the earlier data set is more concentrated than the later. This can be verified further by looking at the spoke plot of both data sets. As for the model checking or goodness-of-fit for our model, we use graphical method which is by plotting the von Mises quantile plot for the residuals of respective data set.

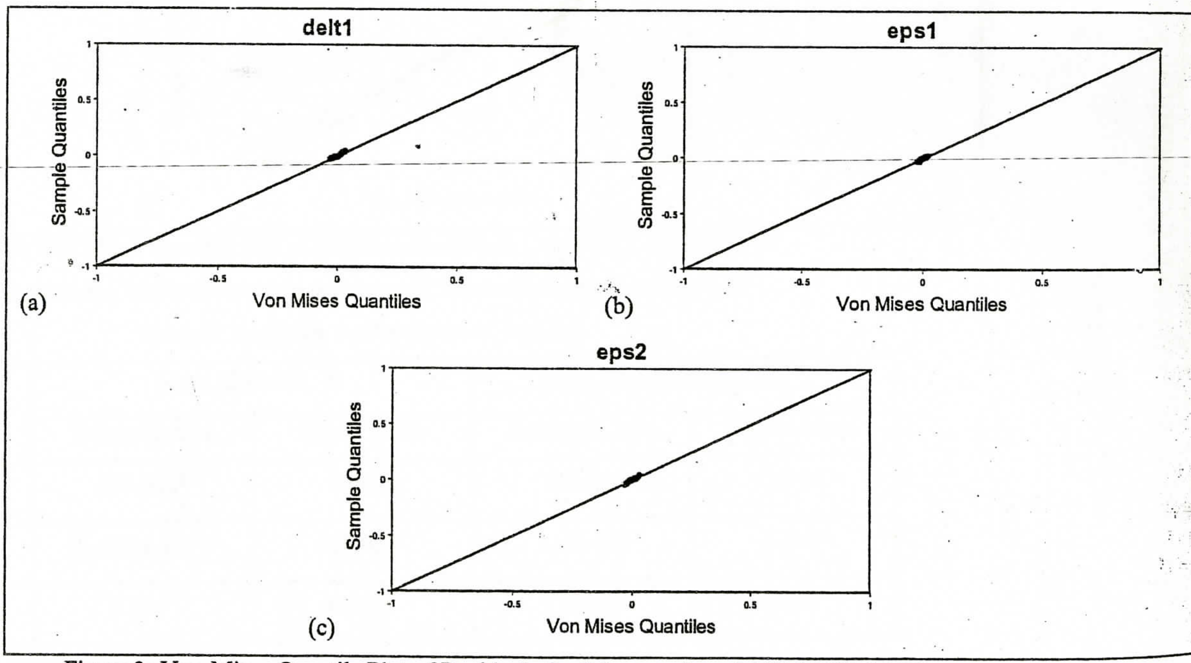

Figure 3: Von Mises Quantile Plot of Residuals for Seberang Jaya (a) X (b) Y1 (c) Y2

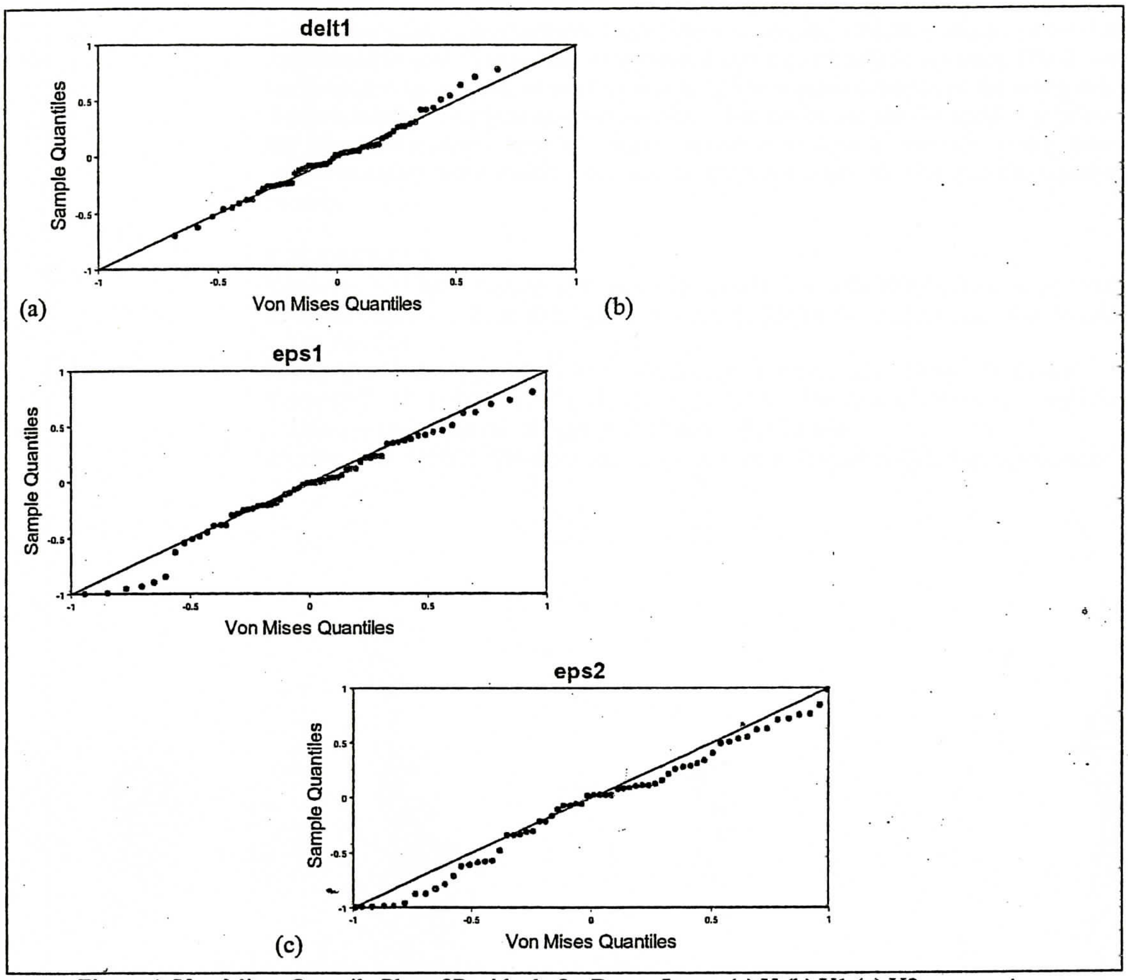

Figure 4: Von Mises Quantile Plot of Residuals for Bayan Lepas (a) X (b) Y1 (c) Y2

The estimated mean direction and concentration parameter of the residuals are shown below:

Table 2: Residuals for the observed data set.

Residuals	Mean direction, $\hat{\mu}$		Concentration parameter, $\hat{\kappa}$	
	Seberang Jaya	Bayan Lepas	Seberang Jaya	Bayan Lepas
δ_i (X)	359.995°	0.003°	790.396	2.940
ε_{1i} (Y1)	(6.48E x 10^{-4})°	359.983°	2145.500	1.541
ε_{2i} (Y2)	0.004°	0.021°	772.508	0.893

From Figure 3 and 4, we can see each residual for Seberang Jaya and Bayan Lepas data set were satisfied the von Mises distribution where the residuals scattered along the line. Table 2 above also showed that the estimated mean direction for each residual is closed to 0 or 2π with concentration parameter, κ. This satisfied our initial assumption that each residual are independently distributed, followed the von Mises distribution with mean direction zero and concentration parameter, κ. Hence, we can say that Seberang Jaya and Bayan Lepas data set were fitted well with the proposed model.

Conclusions

This study proposed the simultaneous linear circular functional relationship model which is an extended form of the linear circular functional relationship model. This model is significant because it allows us to look at the relationship between more than two circular variables simultaneously when only unreplicated data available and all variables are measured inexactly or subject to errors. The above results suggested that the proposed model can be used in assessing the relative calibrations for any given data set in which the measurements are subject to errors. Further, this model can also be applied in other studies which involve the circular variables when our main interest is to look at the relationship between different circular variables rather than predict one variable from the other variable and the variables cannot be recorded exactly.

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