Efficient Approximation for the von Mises Concentration Parameter

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Summary

This paper discusses some of the approximation which can be used to obtain the maximum likelihood estimate of the concentration parameter of the von Mises distribution. We show in this paper that the efficient approximation may also be obtained by solving the equation for the ratio of modified Bessel function of first kind of order one and first kind of order zero. Further, the closed-form solution for parameter concentration is also given. It is found out that the new proposed method performs well especially for large κ.

Keywords: Concentration parameters; von Mises distribution; Bessel function.

1. Introduction

The von Mises distribution is also known as the circular normal distribution and is a continuous probabilty distribution. It may be thought of as the circular version of the normal disribution, since it describes the distribution of a random variate with period 2π. It is used in applications of directional or circular statistics, where a distribution of angles are found which are the result of the addition of many small independent angular deviations, such as target sensing, or grain orientation in a granular material. For a notation, the von Mises distribution is denoted by $M(\mu_0,\kappa)$ where μ_0 $(0 \le \mu_0 < 2\pi)$ is the mean direction, the κ is known as the concentration parameter and has probability density function given by $g(\theta;\mu_0,\kappa)=\{2\pi I_0(\kappa)\}^{-1}\exp\{\kappa\cos(\theta-\mu_0\}$. Further I_0 denotes the modified Bessel function of the first kind and order zero, Mardia & Jupp (2000). The parameters μ_0 and κ can be understood by considering the case where κ is large. The distribution becomes very concentrated about the angle μ_0 with κ being a measure of the concentration. In fact, as κ increases, the distribution approaches a normal distribution in θ with mean μ_0 and variance $\frac{1}{\kappa}$. The probability density can also be expressed as a series of Bessel functions (see Abramowitz and Stegun (1974)). Some of the application of the von Mises distributation is in circular regression model and circular linear functional model which was first introduced by Hussin (2001) and were further discussed by Hussin (2003, 2005).

If $\theta_1,\ldots,\theta_n$ are a random sample from $M(\mu_0,\kappa)$, then the maximum likelihood estimator of κ is $\hat{\kappa}$, the solution of $A(\hat{\kappa})=\overline{R}=\left(\overline{C}^2+\overline{S}^2\right)$, where $\overline{C}=n^{-1}\sum\cos\theta_i$, $\overline{S}=n^{-1}\sum\sin\theta_i$ and $A(x)=I_1(x)/I_0(x)$, where I_1 is the modified Bessel function of the first kind and order one. Thus, $\hat{\kappa}=A^{-1}(\overline{R})$ and limited table of A^{-1} can be found in Mardia (1972). This paper discuss some function which approximate A^{-1} that can be obtained accurately without using the tables, algorithm, interpolation formule or sophisticated computer programme.

2. Approximations for the von Mises concentration parameters

There are several approximations for $A^{-1}(x)$ for all x in $(0,1)$. Amos (1974) proved that

$$\frac{x}{\frac{1}{2}+\left(x^2+\frac{9}{4}\right)^{2}} < A(x) < \frac{x}{\frac{1}{2}+\left(x^2+\frac{1}{4}\right)^{\frac{1}{2}}}, \text{ for } x \geq 0$$

and hence $A^{-1}(x)$ is approximately given by

$$f(x) = \frac{x}{1-x^2}\left[\frac{1}{2}+\left\{1.46\left(1-x^2\right)+\frac{1}{4}\right\}^{\frac{1}{2}}\right].$$

Mardia and Zemroch (1975) give a computer algorithm for calculating $A^{-1}(x)$ together with the tables which is obtained iteratively. Meanwhile, by using the power series for the Bessel function $I_0(x)$ and $I_1(x)$, Dobson (1978) give the approximation of $A^{-1}(x)$ and is gven by

$$f(x) = \begin{cases} 2x + x^3 + \dfrac{5x^5}{6}, & x < 0.65 \\ \dfrac{9-8x+3x^2}{8(1-x)}, & x \geq 0.65 \end{cases}$$

and shown that this approximation give less maximum relative error compared to Amos (1974). Further, Best & Fisher (1981) gives an improved approximation for $A^{-1}(x)$ which is

$$f(x) = \begin{cases} 2x + x^3 + \dfrac{5x^5}{6}, & x < 0.53 \\ -0.4 + 1.39x + \dfrac{0.43}{1-x}, & 0.53 \leq x < 0.85 \\ \dfrac{1}{x^3 - 4x^2 + 3x}, & x \geq 0.85 \end{cases}$$

It's tabulated values can be found in Fisher (1993). In the following section we propose the efficient aproximation by solving the equation for the ratio of modified Bessel function of first kind of order one and first kind of order zero.

3. Approximation based on the modified Bessel function

By defination, $A(\hat{\kappa}) = \left(\overline{C}^2 + \overline{S}^2\right)^{\frac{1}{2}} = t$ (1)

and from the power series for the Bessel function $I_0(x)$ and $I_1(x)$, together with the asymptotic power series

$$A(x) = \frac{I_1(x)}{I_0(x)} \approx 1 - \frac{1}{2x} - \frac{1}{8x^2} - \frac{1}{8x^3} \qquad (2)$$

Solving (1) and (2) to get

$$(8t-8)\hat{\kappa}^3 + 4\hat{\kappa}^2 + \hat{\kappa} + 1 = 0$$

$$\text{or } a_3\hat{\kappa}^3 + a_2\hat{\kappa}^2 + a_1\hat{\kappa} + a_0 = 0 \qquad (3)$$

where $a_3 = 8t - 8$, $a_2 = 4$, $a_1 = 1$ and $a_0 = 1$.

By taking transformation of $\hat{\kappa} = \left(y - \frac{a_2}{3a_3}\right)$, equation (3) can be written as

$$y^3 + py + q = 0 \qquad (4)$$

where $p = \frac{3a_3a_1 - a_2^{\ 2}}{3a_3^{\ 2}} = \frac{3(t-1)-2}{24(t-1)^2}$

and $q = \frac{2a_2^3 - 9a_1a_2a_3 + 27a_0a_3^{\ 2}}{27a_3^{\ 3}} = \frac{4 - 9(t-1) + 54(t-1)^2}{432(t-1)^3}$.

Following Rades & Westergren (1988), the roots for equation (4) are of one real root and two complex roots and are given by

$$y_1 = u + v,$$
$$y_2 = -\left(\frac{u+v}{2}\right) + \left(\frac{u-v}{2}\right)\sqrt{3}i, \text{ and} \qquad (5)$$
$$y_3 = -\left(\frac{u+v}{2}\right) - \left(\frac{u-v}{2}\right)\sqrt{3}i.$$

where

$$u = \left(\frac{-q}{2} + \sqrt{D}\right)^{1/3}, \quad v = \left(\frac{-q}{2} - \sqrt{D}\right)^{1/3} \text{ and } D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 .$$

Hence, the approximation for $\hat{\kappa}$ is given by

$$\hat{\kappa} = \left(-\frac{q}{2} + \sqrt{D}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{D}\right)^{\frac{1}{3}} - \frac{1}{6(t-1)} . \qquad (6)$$

This have been verified further by simulation using the finding roots of polynomial subroutine in SPLUS and we found that the roots as in equation (5) and roots of polynomial subroutine in SPLUS gave very similar results. In the following section, we shown that this approximation give more efficient estimation for $\hat{\kappa}$ compare to the approximations given by Dobson (1978) and Best & Fisher (1981).

4. Simulation results

Computer programs were written using SPLUS language to carry out the simulation study to assess the efficiency of the three different methods of approximating the concentration parameter κ. Circular samples of length $n = 100$ were generated from von Mises distribution with mean 0 and κ= 2, 4, 6, 8, 10, 12 14 and 16.

Let s be the number of simulations and the following computation were obtained from the simulation study.

i) Mean, $\bar{\hat{\kappa}} = \frac{1}{s}\sum \hat{\kappa}_j$,

ii) Estimated Bias = $\bar{\hat{\kappa}} - \kappa$,

iii) Absolute Relative Estimated Bias (%) = $\left(\frac{\left|\bar{\hat{\kappa}} - \kappa\right|}{\kappa}\right) \times 100\%$,

iv) Estimated Standard Errors = $\sqrt{\frac{1}{s-1}\sum\left(\hat{\kappa}_j - \bar{\hat{\kappa}}\right)^2}$,

v) Estimated Root Mean Square Errors (RMSE) = $\sqrt{\frac{1}{s}\sum\left(\hat{\kappa}_j - \kappa\right)^2}$.

vi) Relative Efficiency = $\frac{\text{Estimeted RMSE of Dobson or New}}{\text{Estimated RMSE of Fisher}}$

The simulation results with $s = 5{,}000$ for various true value of concentration parameter are shown in Table 1. The values of mean, estimated bias, absolute relative bias, estimated standard error and RMSE are computed for the Dobson's method, Fisher's method and new proposed method and given in rows 1-15 of Table 1. We also compare the efficiency of Dobson and new proposed methods relative to the Fisher method and the values are given in row 16-18.

It appears from rows 1 to 3 that the mean estimate obtained from the new proposed method is very close to the true κ value compared to the estimate obtained by Dobson and Fisher approximation method. It can be seen clearer from rows 4 to 6 as the biases of estimates for new proposed method are closer to zero. Note also that biases are an increasing function of true κ. Further, it is also clear that, for $\kappa \geq 6$, the absolute relative estimated bias, estimated standard errors and estimated root mean square errors are the smallest for the new proposed method and followed by the Fisher's method as given in row 7-15. However, for smaller κ, Dobson's method has the smallest estimated standard errors and estimated root mean square errors. To make a direct comparison between these approximation methods, the relative efficiency of Dobson's method and new proposed method relative to Fisher's method were computed and results are given in row 16-18. Result in row 18 shows that new proposed method has relative efficiency smaller than one for all values of true κ. These indicate new proposed method is relatively better than Fisher approximation method especially for small κ. New proposed method is also found to be relatively better than the Dobson's method except for small κ.

Conclusion

In this paper, the objective of our study was to evaluate the performance of three different approximating methods of concentration parameter κ, namely, the Dobson's method, Fisher's method and new proposed method through simulation study. Generally, it appears that, for large κ, the new proposed method have a better performance than the Dobson's and Fisher's methods. For smaller κ, new proposed method has the least biases and absolute relative biases but Dobson's method is more efficient with smaller values of estimated standard error and RMSE. Hence, it seems that new proposed method is superior for large κ. However, new proposed method and Dobson's method could be used for small κ.

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	True Value	$\kappa = 2$	$\kappa = 4$	$\kappa = 6$	$\kappa = 8$	$\kappa = 10$	$\kappa = 12$	$\kappa = 14$	$\kappa = 16$
Mean	$\hat{\kappa}^{Dobson}$	2.0571	4.0795	6.177991	8.242157	10.3176	12.38318	14.4343	16.5612
	$\hat{\kappa}^{Fisher}$	2.0348	4.0884	6.151683	8.254001	10.2998	12.37513	14.4193	16.5132
	$\hat{\kappa}^{new}$	2.0152	4.0634	6.154961	8.183872	10.2836	12.36134	14.4093	16.4854
Est Bias	$\hat{\kappa}^{Dobson}$	0.0571	0.0795	0.177991	0.242157	0.3176	0.3831806	0.4343	0.5612
	$\hat{\kappa}^{Fisher}$	0.0348	0.0884	0.1516833	0.2540009	0.2998	0.3751307	0.4193	0.5132
	$\hat{\kappa}^{new}$	0.0152	0.0634	0.1549606	0.1838723	0.2836	0.3613394	0.4093	0.4854
Absolute Relative Bias (%)	$\hat{\kappa}^{Dobson}$	2.8569	1.9894	2.966517	3.026963	3.1768	3.193172	3.1024	3.5079
	$\hat{\kappa}^{Fisher}$	1.7421	2.2101	2.528055	3.175011	2.9988	3.126089	2.9953	3.2078
	$\hat{\kappa}^{new}$	0.7629	1.5853	2.582677	2.298404	2.8361	3.011162	2.9242	3.0340
Est S.E	$\hat{\kappa}^{Dobson}$	0.2175	0.5467	0.8621339	1.133723	1.4594	1.782754	2.0498	2.3967
	$\hat{\kappa}^{Fisher}$	0.2589	0.5466	0.8528974	1.149091	1.4699	1.758692	2.0598	2.3538
	$\hat{\kappa}^{new}$	0.2305	0.5489	0.8502372	1.131352	1.4621	1.71867	2.0231	2.3467
Est RMSE	$\hat{\kappa}^{Dobson}$	0.0505	0.3052	0.7748071	1.343711	2.2304	3.324403	4.3897	6.0580
	$\hat{\kappa}^{Fisher}$	0.0682	0.3065	0.7502963	1.384663	2.2502	3.233101	4.4178	5.8028
	$\hat{\kappa}^{new}$	0.0533	0.3053	0.7467715	1.313509	2.2180	3.083802	4.2598	5.7416
Efficiency	$\hat{\kappa}^{Dobson}$	0.7408	0.9957	1.032668	0.9704242	0.9912	1.02824	0.9936	1.0439
	$\hat{\kappa}^{Fisher}$	1	1	1	1	1	1	1	1
	$\hat{\kappa}^{new}$	0.7820	0.9960	0.9953022	0.948613	0.9856	0.9538218	0.9642	0.9894

Table 1: Simulation results for various true value of parameter concentration.