Research Article

Some Results on Warped Product Submanifolds of a Sasakian Manifold

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We study warped product Pseudo-slant submanifolds of Sasakian manifolds. We prove a theorem for the existence of warped product submanifolds of a Sasakian manifold in terms of the canonical structure $F$.

1. Introduction

The notion of slant submanifold of almost contact metric manifold was introduced by Lotta [1]. Latter, Cabrerizo et al. investigated slant and semislant submanifolds of a Sasakian manifold and obtained many interesting results [2, 3].

The notion of warped product manifolds was introduced by Bishop and O’Neill in [4]. Latter on, many research articles appeared exploring the existence or nonexistence of warped product submanifolds in different spaces (cf. [5–7]). The study of warped product semislant submanifolds of Kaehler manifolds was introduced by Sahin [8]. Recently, Hasegawa and Mihai proved that warped product of the type $N_1 \times_\lambda N_T$ in Sasakian manifolds is trivial where $N_T$ and $N_L$ are $\phi$–invariant and anti-invariant submanifolds of a Sasakian manifold, respectively [9].

In this paper we study warped product submanifolds of a Sasakian manifold. We will see in this paper that for a warped product of the type $M = N_1 \times_\lambda N_2$, if $N_1$ is any Riemannian submanifold tangent to the structure vector field $\xi$ of a Sasakian manifold $\tilde{M}$ then $N_2$ is an anti-invariant submanifold and if $\xi$ is tangent to $N_2$ then there is no warped product. Also, we will show that the warped product of the type $M = N_1 \times_\lambda N_0$ of a Sasakian manifold $\tilde{M}$ is trivial and that the warped product of the type $N_T \times_\lambda N_L$ exists and obtains a result in terms of canonical structure.
2. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional manifold with almost contact structure $(\phi, \xi, \eta)$ defined by a $(1,1)$ tensor field $\phi$, a vector field $\xi$, and the dual 1–form $\eta$ of $\xi$, satisfying the following properties [10]:

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.$$  \hspace{1cm} (2.1)

There always exists a Riemannian metric $g$ on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition:

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y).$$  \hspace{1cm} (2.2)

An almost contact metric manifold $\tilde{M}$ is called Sasakian if

$$\left(\bar{\nabla}_X \phi \right) Y = g(X,Y)\xi - \eta(Y)X$$  \hspace{1cm} (2.3)

for all $X,Y$ in $T\tilde{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $g$ on $\tilde{M}$. From (2.3), it follows that

$$\bar{\nabla}_X \xi = -\phi X.$$  \hspace{1cm} (2.4)

Let $M$ be submanifold of an almost contact metric manifold $\tilde{M}$ with induced metric $g$ and if $\nabla$ and $\nabla_\perp$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively, then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y),$$  \hspace{1cm} (2.5)

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N,$$  \hspace{1cm} (2.6)

for each $X, Y \in TM$ and $N \in T^\perp M$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$), respectively, for the immersion of $M$ into $\tilde{M}$. They are related as

$$g(h(X,Y), N) = g(A_N X, Y),$$  \hspace{1cm} (2.7)

where $g$ denotes the Riemannian metric on $\tilde{M}$ as well as the one induced on $M$.

For any $X \in TM$, we write

$$\phi X = PX + FX,$$  \hspace{1cm} (2.8)

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$. 

Similarly, for any $N \in T^1M$, we write

$$\phi N = tN + fN,$$  \hspace{1cm} (2.9)

where $tN$ is the tangential component and $fN$ is the normal component of $\phi N$. We shall always consider $\xi$ to be tangent to $M$. The submanifold $M$ is said to be \textit{invariant} if $F$ is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand, $M$ is said to be \textit{anti-invariant} if $P$ is identically zero, that is, $\phi X \in T^1M$, for any $X \in TM$.

For each nonzero vector $X$ tangent to $M$ at $x$, such that $X$ is not proportional to $\xi$, we denote by $\theta(X)$ the angle between $\phi X$ and $PX$.

$M$ is said to be slant \cite{3} if the angle $\theta(X)$ is constant for all $X \in TM - \{\xi\}$ and $x \in M$. The angle $\theta$ is called slant angle \textit{or Wirtinger angle}. Obviously, if $\theta = 0$, $M$ is invariant and if $\theta = \pi/2$, $M$ is an anti-invariant submanifold. If the slant angle of $M$ is different from 0 and $\pi/2$ then it is called \textit{proper slant}.

A characterization of slant submanifolds is given by the following.

\textbf{Theorem 2.1 (see \cite{3}).} Let $M$ be a submanifold of an almost contact metric manifold $\overline{M}$, such that $\xi \in TM$. Then $M$ is slant if and only if there exists a constant $\delta \in [0, 1]$ such that

$$P^2 = \delta (-I + \eta \otimes \xi).$$ \hspace{1cm} (2.10)

Furthermore, in such case, if $\theta$ is slant angle, then $\delta = \cos^2 \theta$.

Following relations are straightforward consequences of (2.10)

$$g(PX, PY) = \cos^2 \theta [g(X, Y) - \eta(X) \eta(Y)],$$ \hspace{1cm} (2.11)

$$g(FX, FY) = \sin^2 \theta [g(X, Y) - \eta(X) \eta(Y)]$$ \hspace{1cm} (2.12)

for any $X, Y$ tangent to $M$.

\section{Warped and Doubly Warped Product Manifolds}

Let $(N_1, g_1)$ and $(N_2, g_2)$ be two Riemannian manifolds and $\lambda$ a positive differentiable function on $N_1$. The warped product of $N_1$ and $N_2$ is the Riemannian manifold $N_1 \times_\lambda N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + \lambda^2 g_2.$$ \hspace{1cm} (3.1)

A warped product manifold $N_1 \times_\lambda N_2$ is said to be \textit{trivial} if the warping function $\lambda$ is constant. We recall the following general formula on a warped product \cite{4}:

$$\nabla_X V = \nabla_V X + (X \ln \lambda) V,$$ \hspace{1cm} (3.2)

where $X$ is tangent to $N_1$ and $V$ is tangent to $N_2$. 
Let $M = N_1 \times N_2$ be a warped product manifold then $N_1$ is totally geodesic and $N_2$ is totally umbilical submanifold of $M$, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by Unal [11]. A doubly warped product manifold of $N_1$ and $N_2$, denoted as $f_2 N_1 \times f_1 N_2$ is the manifold $N_1 \times N_2$ endowed with a metric $g$ defined as

$$g = f_2^2 g_1 + f_1^2 g_2$$

(3.3)

where $f_1$ and $f_2$ are positive differentiable functions on $N_1$ and $N_2$, respectively.

In this case formula (3.2) is generalized as

$$\nabla_X Z = (X \ln f_1) Z + (Z \ln f_2) X$$

(3.4)

for each $X$ in $TN_1$ and $Z$ in $TN_2$ [7].

If neither $f_1$ nor $f_2$ is constant we have a nontrivial doubly warped product $M = f_2 N_1 \times f_1 N_2$. Obviously in this case both $N_1$ and $N_2$ are totally umbilical submanifolds of $M$.

Now, we consider a doubly warped product of two Riemannian manifolds $N_1$ and $N_2$ embedded into a Sasakian manifold $\overline{M}$ such that the structure vector field $\xi$ is tangent to the submanifold $M = f_2 N_1 \times f_1 N_2$. Consider $\xi$ is tangent to $N_1$, then for any $V \in TN_2$ we have

$$\nabla_V \xi = (\xi \ln f_1) V + (V \ln f_2) \xi + h(V, \xi) = -PV - FV.$$  

(3.5)

Thus from (2.4), (2.5), (2.8), and (3.5), we get

$$\overline{\nabla}_V \xi = (\xi \ln f_1) V + (V \ln f_2) \xi + h(V, \xi) = -PV - FV.$$  

(3.6)

On comparing tangential and normal parts and using the fact that $\xi$, $V$, and $PV$ are mutually orthogonal vector fields, (3.6) implies that

$$V \ln f_2 = 0, \quad \xi \ln f_1 = 0, \quad h(V, \xi) = -FV, \quad PV = 0.$$  

(3.7)

This shows that $f_2$ is constant and $N_2$ is an anti-invariant submanifold of $\overline{M}$, if the structure vector field $\xi$ is tangent to $N_1$.

Similarly, if $\xi$ is tangent to $N_2$ and for any $U \in TN_1$ we have

$$\overline{\nabla}_U \xi = (\xi \ln f_2) U + (U \ln f_1) \xi + h(U, \xi) = -PU - FU,$$

(3.8)

which gives

$$U \ln f_1 = 0, \quad \xi \ln f_2 = 0, \quad PU = 0, \quad h(U, \xi) = -FU.$$  

(3.9)

That is, $f_1$ is constant and $N_1$ is an anti-invariant submanifold of $\overline{M}$. 
Note 1. From the above conclusion we see that for warped product submanifolds $M = N_1 \times N_2$ of a Sasakian manifold $\overline{M}$, if the structure vector field $\xi$ is tangent to the first factor $N_1$ then second factor $N_2$ is an anti-invariant submanifold. On the other hand the warped product $M = N_1 \times N_2$ is trivial if the structure vector field $\xi$ is tangent to $N_2$.

To study the warped product submanifolds $N_1 \times N_2$ with structure vector field $\xi$ tangent to $N_1$, we have obtained the following lemma.

Lemma 3.1 (see [12]). Let $M = N_1 \times N_2$ be a proper warped product submanifold of a Sasakian manifold $\overline{M}$, with $\xi \in TN_1$, where $N_1$ and $N_2$ are any Riemannian submanifolds of $\overline{M}$. Then

(i) $\xi \ln \lambda = 0$,

(ii) $A_{FZ}X = -th(X, Z)$,

(iii) $g(h(X, Z), FY) = g(h(X, Y), FZ)$,

(iv) $g(h(X, Z), FW) = g(h(X, W), FZ)$

for any $X, Y \in TN_1$ and $Z, W \in TN_2$.

4. Warped Product Pseudoslant Submanifolds

The study of semislant submanifolds of almost contact metric manifolds was introduced by Cabrero et.al. [2]. A semislant submanifold $M$ of an almost contact metric manifold $\overline{M}$ is a submanifold which admits two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^\theta$ such that $\mathcal{D}$ is invariant under $\phi$ and $\mathcal{D}^\theta$ is slant with slant angle $\theta \neq 0$, that is, $\phi \mathcal{D} = \mathcal{D}$ and $\phi Z$ makes a constant angle $\theta$ with $TM$ for each $Z \in \mathcal{D}^\theta$. In particular, if $\theta = \pi/2$, then a semislant submanifold reduces to a contact CR-submanifold. For a semislant submanifold $M$ of an almost contact metric manifold, we have

$$TM = \mathcal{D} \oplus \mathcal{D}^\theta \oplus \{\xi\}. \quad (4.1)$$

Similarly we say that $M$ is an pseudo-slant submanifold of $\overline{M}$ if $\mathcal{D}$ is an anti-invariant distribution of $M$, that is, $\phi \mathcal{D} \subseteq T^\perp M$ and $\mathcal{D}^\theta$ is slant with slant angle $\theta \neq 0$. The normal bundle $T^\perp M$ of an pseudo-slant submanifold is decomposed as

$$T^\perp M = FT M \oplus \mu, \quad (4.2)$$

where $\mu$ is an invariant subbundle of $T^\perp M$.

From the above note, we see that for warped product submanifolds $N_1 \times N_2$ of a Sasakian manifold $\overline{M}$, one of the factors is an anti-invariant submanifold of $\overline{M}$. Thus, if the manifolds $N_\theta$ and $N_\perp$ are slant and anti-invariant submanifolds of Sasakian manifold $\overline{M}$, then their possible warped product pseudo-slant submanifolds may be given by one of the following forms:

(a) $N_\perp \times N_\theta$,

(b) $N_\theta \times N_\perp$. 
Comparing tangential and normal parts, we get

\[ N \theta \times N_\theta \text{ and } N_\theta \times N_\theta \text{ when } \xi \text{ is in } TN_\perp \text{ and in } TN_\theta, \text{ respectively.} \]

For the warped product of the type (a), we have

**Theorem 4.1.** There do not exist the warped product Pseudo-slant submanifolds \( M = N_\perp \times N_\theta \) where \( N_\perp \) is an anti-invariant and \( N_\theta \) is a proper slant submanifold of a Sasakian manifold \( M \) such that \( \xi \) is tangent to \( N_\perp \).

**Proof.** For any \( X \in TN_\theta \) and \( Z \in TN_\perp \), we have

\[
\left( \nabla_X \phi \right) Z = \nabla_X \phi Z - \phi \nabla_X Z. \tag{4.3}
\]

Using (2.3), (2.5), (2.6), and the fact that \( \xi \) is tangent to \( N_\perp \), we obtain

\[
-\eta(Z)X = -A_{FZ}X + \nabla_X^\perp FZ - P \nabla_X Z - F \nabla_X Z - th(X, Z) - fh(X, Z). \tag{4.4}
\]

Comparing tangential and normal parts, we get

\[
\eta(Z)X = A_{FZ}X + P \nabla_X Z + th(X, Z) \tag{4.5}
\]

Equation (4.5) takes the form on using (3.2) as

\[
\eta(Z)X = A_{FZ}X + (Z \ln \lambda)PX + th(X, Z). \tag{4.6}
\]

Taking product with \( PX \), the left hand side of the above equation is zero using the fact that \( X \) and \( PX \) are mutually orthogonal vector fields. Then

\[
0 = g(A_{FZ}X, PX) + (Z \ln \lambda)g(PX, PX) + g(th(X, Z), PX). \tag{4.7}
\]

Using (2.7), (2.11) and the fact that \( \xi \) is tangent to \( N_\perp \), we get

\[
(Z \ln \lambda)\cos^2 \theta \|X\|^2 = g(h(X, Z), FPX) - g(h(X, PX), FX). \tag{4.8}
\]

As \( \theta \neq \pi/2 \), then interchanging \( X \) by \( PX \) and taking account of (2.10), we obtain

\[
(Z \ln \lambda)\cos^4 \theta \|X\|^2 = -\cos^2 \theta g(h(PX, Z), FX) + \cos^2 \theta g(h(X, PX), FZ) \tag{4.9}
\]

or

\[
(Z \ln \lambda)\cos^2 \theta \|X\|^2 = g(h(X, PX), FZ) - g(h(PX, Z), FX). \tag{4.10}
\]
Adding equations (4.8) and (4.10), we get

$$2(Z\ln \lambda)\cos^2\theta ||X||^2 = g(h(X, Z), FPX) - g(h(PX, Z), FX).$$

(4.11)

The right hand side of the above equation is zero by Lemma 3.1(iv); then

$$(Z\ln \lambda)\cos^2\theta ||X||^2 = 0.$$  \hspace{1cm} (4.12)

Since $N_\theta$ is proper slant and $X$ is nonnull, then

$$Z\ln \lambda = 0.$$  \hspace{1cm} (4.13)

In particular, for $Z = \xi \in T N_\perp$, Lemma 3.1 (i) implies that $\xi \ln \lambda = 0$. This means that $\lambda$ is constant on $N_\perp$. Hence the theorem is proved. \[\Box\]

Now, the other case is dealt with in the following theorem.

**Theorem 4.2.** Let $M = N_T \times \lambda N_\perp$ be a warped product submanifold of a Sasakian manifold $\overline{M}$ such that $N_T$ is an invariant submanifold tangent to $\xi$ and $N_\perp$ is an anti-invariant submanifold of $\overline{M}$. Then $(\overline{\nabla}_X F)Z$ lies in the invariant normal subbundle for each $X \in T N_T$ and $Z \in T N_\perp$.

**Proof.** As $M = N_T \times \lambda N_\perp$ is a warped product submanifold with $\xi$ tangent to $N_T$, then by (2.3),

$$\left(\overline{\nabla}_X \phi\right)Z = 0,$$

(4.14)

for any $X \in T N_T$ and $Z \in T N_\perp$. Using this fact in the formula

$$\left(\overline{\nabla}_U \phi\right)V = \overline{\nabla}_U \phi V - \phi \overline{\nabla}_U V$$

(4.15)

for each $U, V \in T \overline{M}$, thus, we obtain

$$\overline{\nabla}_X \phi Z = \phi \overline{\nabla}_X Z.$$  \hspace{1cm} (4.16)

Then from (2.5) and (2.6), we get

$$-A_{FZ}X + \nabla^\perp_X FZ = \phi(\nabla_X Z + h(X, Z)).$$

(4.17)

Which on using (2.8) and (2.9) yields

$$-A_{FZ}X + \nabla^\perp_X FZ = P \nabla_X Z + F \nabla_X Z + th(X, Z) + fh(X, Z).$$

(4.18)

From the normal components of the above equation, formula (3.2) gives

$$\nabla^\perp_X FZ = (\lambda \ln \lambda)FZ + fh(X, Z).$$

(4.19)
Taking the product in (4.19) with $FW_1$ for any $W_1 \in TN_L$, we get

$$g\left(\nabla^\perp_X FZ, FW_1\right) = (X \ln \lambda) g(FZ, FW_1) + g(fh(X, Z), FW_1)$$  \hspace{1cm} (4.20)

or

$$g\left(\nabla^\perp_X FZ, FW_1\right) = (X \ln \lambda) g(\phi Z, \phi W_1) + g(\phi h(X, Z), \phi W_1).$$  \hspace{1cm} (4.21)

Then from (2.2), we have

$$g\left(\nabla^\perp_X FZ, FW_1\right) = (X \ln \lambda) g(Z, W_1).$$  \hspace{1cm} (4.22)

On the other hand, we have

$$\left(\nabla_X F\right)Z = \nabla^\perp_X FZ - F \nabla_X Z.$$  \hspace{1cm} (4.23)

Taking the product in (4.23) with $FW_1$ for any $W_1 \in TN_L$ and using (4.22), (2.2), (3.2), and the fact that $\xi$ is tangential to $N_T$, we obtain that

$$g\left(\left(\nabla_X F\right)Z, FW_1\right) = 0,$$  \hspace{1cm} (4.24)

for any $X \in TN_T$ and $Z, W_1 \in TN_L$.

Now, if $W_2 \in TN_T$ then using the formula (4.23), we get

$$g\left(\left(\nabla_X F\right)Z, \phi W_2\right) = g\left(\nabla^\perp_X FZ, \phi W_2\right) - g(F \nabla_X Z, \phi W_2).$$  \hspace{1cm} (4.25)

As $N_T$ is an invariant submanifold, then $\phi W_2 \in TN_T$ for any $W_2 \in TN_T$, thus using the fact that the product of tangential component with normal is zero, we obtain that

$$g\left(\left(\nabla_X F\right)Z, \phi W_2\right) = 0,$$  \hspace{1cm} (4.26)

for any $X, W_2 \in TN_T$ and $Z \in TN_L$. Thus from (4.24) and (4.26), it follows that $\left(\nabla_X F\right)Z \in \mu$. Thus the proof is complete.

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