IJMMS 2003:59, 3761–3767
PII. S0161171203302339
http://ijmms.hindawi.com

COEFFICIENTS ESTIMATES FOR FUNCTIONS IN $B_n(\alpha)$

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Received 12 February 2003

We consider functions f, analytic in the unit disc and of the normalised form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For functions $f \in B_n(\alpha)$, the class of functions involving the Sălăgean differential operator, we give some coefficient estimates, namely, $|a_2|$, $|a_3|$, and $|a_4|$.

2000 Mathematics Subject Classification: 30C45.

1. Introduction. Let A be the class of functions f which are analytic in the unit disc $D = \{z : |z| < 1\}$ and are of the form

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j. \tag{1.1}$$

For functions $f \in A$, we introduce the subclass $B_n(\alpha)$ given by the following definition.

Definition 1.1. For $\alpha > 0$ and $n = 0, 1, 2, \ldots$, a function f normalised by (1.1) belongs to $B_n(\alpha)$ if and only if, for $z \in D$,

$$\operatorname{Re} \frac{D^n [f(z)]^\alpha}{z^\alpha} > 0, \tag{1.2}$$

where D^n denotes the differential operator with $D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$ and $D^\circ f(z) = f(z)$.

Remark 1.2. The differential operator D^n was introduced by Sălăgean [5].

For $n = 1$, $B_1(\alpha)$ denotes the class of Bazilević functions with logarithmic growth studied [4, 6, 7], amongst others. In [2], the author established some properties of the class $B_n(\alpha)$ including showing that $B_n(\alpha)$ forms a subclass of S, the class of all analytic, normalized, and univalent functions in D. The class $B_0(\alpha)$ was initiated by Yamaguchi [8].

2. Preliminary results. In proving our results, we need the following lemmas. However, we first denote P to be the class of analytic functions with a positive real part in D.

Lemma 2.1. *Let $p \in P$ and let it be of the form $p(z) = 1 + \sum_{i=1}^{\infty} c_i z^i$. Then*
(i) $|c_i| \le 2$ *for* $i \ge 1$,
(ii) $|c_2 - \mu c_1^2| \le 2\max\{1, |1-2\mu|\}$ *for any* $\mu \in \mathbb{C}$.

Lemma 2.2 (see [3]). *If the functions $1 + \sum_{v=1}^{\infty} b_v z^v$ and $1 + \sum_{v=1}^{\infty} c_v z^v$ belong to P, then the same is true for the function $1 + (1/2)\sum_{v=1}^{\infty} b_v c_v z^v$.*

Lemma 2.3 (see [3]). *Let $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$ and let $1 + g(z) = 1 + g_1 z + g_2 z^2 + \cdots$ be functions in P. Set $\gamma_0 = 1$ and for $v \ge 1$,*

$$\gamma_v = 2^{-v}\left[1 + \frac{1}{2}\sum_{\mu=1}^{v}\binom{v}{\mu} h_\mu\right]. \tag{2.1}$$

If A_k is defined by

$$\sum_{v=1}^{\infty} (-1)^{v+1}\gamma_{v-1}(g(z))^v = \sum_{k=1}^{\infty} A_k z^k, \tag{2.2}$$

then

$$|A_k| \le 2. \tag{2.3}$$

3. Results

Theorem 3.1. *If $\alpha > 0$, $n = 0, 1, 2, \dots$, and $f \in B_n(\alpha)$ (n is fixed) with $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then the following inequalities hold:*

$$|a_2| \le \frac{2\alpha^{n-1}}{(1+\alpha)^n}, \tag{3.1}$$

$$|a_3| \le \begin{cases} \dfrac{2\alpha^{n-1}}{(2+\alpha)^n}\left(1 - \left(\dfrac{\alpha-1}{\alpha}\right)\left(\dfrac{\alpha^2+2\alpha}{\alpha^2+2\alpha+1}\right)^n\right), & \text{for } 0 < \alpha < 1, \\ \dfrac{2\alpha^{n-1}}{(2+\alpha)^n}, & \text{for } \alpha \ge 1, \end{cases} \tag{3.2}$$

$$|a_4| \le \begin{cases} \dfrac{2\alpha^{n-1}}{(3+\alpha)^n} \\ \qquad + \dfrac{4(1-\alpha)\alpha^{2n-2}}{(1+\alpha)^n(2+\alpha)^n}\left(1 + \dfrac{(1-2\alpha)(2+\alpha)^n\alpha^{n-1}}{3(1+\alpha)^{2n}}\right), & \text{for } 0 < \alpha < 1, \\ \dfrac{2\alpha^{n-1}}{(3+\alpha)^n}, & \text{for } \alpha \ge 1. \end{cases} \tag{3.3}$$

Remark 3.2. When $n = 1$, the above results reduce to those obtained by Singh [6].

Proof. For $f \in B_n(\alpha)$, Definition 1.1 gives

$$\operatorname{Re}\frac{D^n f(z)^\alpha}{z^\alpha} > 0. \tag{3.4}$$

Inequality (3.4) suggests that there exists $p \in P$ such that for $z \in D$,

$$\frac{D^n f(z)^\alpha}{z^\alpha} = \alpha^n p(z). \tag{3.5}$$

Next, writing $D^n f(z)^\alpha$ as $z[D^{n-1}f(z)^\alpha]'$ and $p(z) = 1+\sum_{i=1}^{\infty} c_i z^i$ in (3.5), it follows that

$$[D^{n-1}f(z)^\alpha]' = \alpha^n\left(z^{\alpha-1}+\sum_{i=1}^{\infty} c_i z^{i+\alpha-1}\right) \tag{3.6}$$

and integration gives

$$\frac{D^{n-1}f(z)^\alpha}{z^\alpha} = \alpha^{n-1}\left[1+\sum_{i=1}^{\infty}\alpha\frac{c_i z^i}{(i+\alpha)}\right]. \tag{3.7}$$

Now, repeating the process, we are able to establish the following relation which holds in general for any $k = 0,1,2,\ldots,n$

$$\frac{D^{n-k}f(z)^\alpha}{z^\alpha} = \alpha^{n-k}\left[1+\sum_{i=1}^{\infty}\alpha^k\frac{c_i z^i}{(i+\alpha)^k}\right]. \tag{3.8}$$

In particular, when $n = k$, we have

$$\frac{D^0 f(z)^\alpha}{z^\alpha} = \left(\frac{f(z)}{z}\right)^\alpha = 1+\sum_{i=1}^{\infty}\alpha^n\frac{c_i z^i}{(i+\alpha)^n}. \tag{3.9}$$

On comparing coefficients in (3.9) with $f(z) = z+\sum_{j=2}^{\infty} a_j z^j$, we obtain

$$\alpha a_2 = \frac{\alpha^n c_1}{(1+\alpha)^n}, \tag{3.10}$$

$$\alpha a_3 = \frac{\alpha^n c_2}{(2+\alpha)^n}+\frac{\alpha(1-\alpha)a_2^2}{2}, \tag{3.11}$$

$$\alpha a_4 = \frac{\alpha^n c_3}{(3+\alpha)^n}+\frac{\alpha(1-\alpha)(\alpha-2)a_2^3}{6}+\alpha(1-\alpha)a_3a_2. \tag{3.12}$$

Inequality (3.1) follows easily from (3.10) for all $\alpha > 0$ since $|c_1| \le 2$.

Eliminating a_2 in (3.11), we have

$$\begin{aligned} a_3 &= \frac{\alpha^{n-1}c_2}{(2+\alpha)^n} + \frac{(1-\alpha)}{2}\left(\frac{\alpha^{n-1}c_1}{(1+\alpha)^n}\right)^2 \\ &= \frac{\alpha^{n-1}}{(2+\alpha)^n}\left[c_2 - \frac{(\alpha-1)}{2}\frac{(2+\alpha)^n}{(1+\alpha)^{2n}}\alpha^{n-1}c_1^2\right] \\ &= \frac{\alpha^{n-1}}{(2+\alpha)^n}(c_2-\mu c_1^2) \\ &\le \frac{2\alpha^{n-1}}{(2+\alpha)^n}\max\{1, |1-2\mu|\}, \end{aligned} \tag{3.13}$$

where we used Lemma 2.1(ii) with

$$2\mu = \frac{(\alpha-1)\alpha^{n-1}}{(1+\alpha)^n}\left(\frac{2+\alpha}{1+\alpha}\right)^n. \tag{3.14}$$

Since $\mu \ge 0$ for $\alpha \ge 1$, both inequalities in (3.2) are easily obtained.

We now prove (3.3). Using (3.10) and (3.11) in (3.12) gives

$$a_4 = \frac{\alpha^{n-1}}{(3+\alpha)^n}\left[c_3 + \frac{(1-\alpha)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n}\left(\frac{c_1c_2}{(2+\alpha)^n} + \frac{(1-2\alpha)\alpha^{n-1}c_1^3}{6(1+\alpha)^{2n}}\right)\right]. \tag{3.15}$$

First, we consider the case $0 < \alpha < 1/2$. Applying the triangle inequality with Lemma 2.1(i) in (3.15) results in the inequality

$$|a_4| \le \frac{2\alpha^{n-1}}{(3+\alpha)^n}\left[1 + \frac{2(1-\alpha)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n}\left(\frac{1}{(2+\alpha)^n} + \frac{(1-2\alpha)\alpha^{n-1}}{3(1+\alpha)^{2n}}\right)\right] \tag{3.16}$$

which is the first inequality in (3.3).

For the case $1/2 \le \alpha < 1$, we use Carathéodory-Toeplitz result which states that for some ε with $|\varepsilon| < 1$,

$$c_2 = \frac{c_1^2}{2} + \varepsilon\left(2 - \frac{|c_1|^2}{2}\right). \tag{3.17}$$

Thus, (3.15) becomes

$$\begin{aligned} a_4 = \frac{\alpha^{n-1}}{(3+\alpha)^n}\Bigg[c_3 &+ \frac{(1-\alpha)(3+\alpha)^n\alpha^{n-1}c_1}{(1+\alpha)^n} \\ &\times\left(\frac{c_1^2}{2(2+\alpha)^n} + \frac{(1-2\alpha)\alpha^{n-1}c_1^2}{6(1+\alpha)^{2n}} + \frac{\varepsilon}{(2+\alpha)^n}\left(\frac{2-|c_1|^2}{2}\right)\right)\Bigg]. \end{aligned} \tag{3.18}$$

We then have

$$|a_4| \le \frac{\alpha^{n-1}}{(3+\alpha)^n}\left(|c_3| + \frac{(1-\alpha)(3+\alpha)^n\alpha^{n-1}|c_1|}{(1+\alpha)^n(2+\alpha)^n}\left|\frac{c_1^2}{2}w - \frac{|c_1|^2}{2}\varepsilon + 2\varepsilon\right|\right), \tag{3.19}$$

where

$$w = 1 + \frac{(1-2\alpha)\alpha^{n-1}(2+\alpha)^n}{3(1+\alpha)^{2n}}. \tag{3.20}$$

Since $0 < w \le 1$ and $|\varepsilon| < 1$, it is easily shown that

$$|a_4| \le \frac{\alpha^{n-1}}{(3+\alpha)^n}\left(|c_3| + \frac{(1-\alpha)(3+\alpha)^n\alpha^{n-1}|c_1|}{(1+\alpha)^n(2+\alpha)^n}\left(\frac{|c_1|^2}{2}(w-1)+2\right)\right) \tag{3.21}$$

and the result follows trivially when using $|c_1| \le 2$ and $|c_3| \le 2$.

Finally, we consider (3.3) for the case $\alpha \ge 1$. Here, we use a method introduced by Nehari and Netanyahu [3] which was also used by Singh [6] and the author in [1].

First, let h and g be defined as in Lemma 2.3, and since $p \in P$, Lemma 2.2 indicates that

$$1 + G(z) = 1 + \frac{1}{2}\sum_{k=1}^{\infty} g_k c_k z^k \tag{3.22}$$

also belongs to P.

Next, it follows from (2.2) that, with g replaced by G,

$$|A_3| = \left|\frac{1}{2}g_3c_3 - \frac{1}{2}\gamma_1 g_1 g_2 c_1 c_2 + \frac{1}{8}\gamma_2 g_1^3 c_1^3\right|. \tag{3.23}$$

Rewriting (3.15) as

$$\begin{aligned}\alpha^{1-n}(3+\alpha)^n a_4 = c_3 &+ \frac{(1-\alpha)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n}c_1c_2 \\ &+ \frac{(1-\alpha)(1-2\alpha)(3+\alpha)^n\alpha^{2n-2}}{6(1+\alpha)^{3n}}c_1^3\end{aligned} \tag{3.24}$$

and comparing it with (3.23), the required result is easily obtained since, by Lemma 2.3, $|A_3| = ((3+\alpha)^n/(\alpha^{n-1}))|a_4| \le 2$. This however is only true if we can show the existence of functions h and ψ in P where $\psi(z) = 1 + g(z)$. To be simple, we choose $\psi(z) = (1+z)/(1-z)$. Thus, now it remains to construct and show that an $h \in P$.

Now since $g_1 = g_2 = g_3 = 2$, it follows from (3.23) and (3.24) that

$$2\gamma_1 = \frac{(\alpha-1)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n}, \tag{3.25}$$

$$\gamma_2 = \frac{(1-\alpha)(1-2\alpha)(3+\alpha)^n\alpha^{2n-2}}{6(1+\alpha)^{3n}}. \tag{3.26}$$

However, from (2.1), we have

$$\gamma_1 = \frac{1}{2}\left(1+\frac{1}{2}h_1\right), \tag{3.27}$$

$$\gamma_2 = \frac{1}{4}\left(1+h_1+\frac{1}{2}h_2\right). \tag{3.28}$$

Solving for h_1 by eliminating γ_1 from (3.25) and (3.27), we obtain

$$|h_1| = 2\left|\frac{(\alpha-1)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n}-1\right|. \tag{3.29}$$

Quite trivially, it can be seen that $|h_1| \le 2$ for $\alpha \ge 1$.

In a similar manner, eliminating γ_2 from (3.26) and (3.28) and using h_1 given by (3.29), we have

$$h_2 = 2\left\{1-\frac{2}{3}\left(1-\frac{1}{\alpha}\right)\left(\frac{\alpha^2+3\alpha}{\alpha^2+3\alpha+2}\right)^n\left[\left(\frac{1-2\alpha}{\alpha}\right)\left(\frac{\alpha^2+2\alpha}{\alpha^2+2\alpha+1}\right)^n+3\right]\right\}. \tag{3.30}$$

For $\alpha \ge 1$, elementary calculations show that $|h_2| \le 2$.

Next, we construct h by first setting it to be of the form

$$h(z) = \frac{\mu_1(1-z)}{1+z}+\frac{\mu_2(1+\lambda z^2)}{1-\lambda z^2} \tag{3.31}$$

with

$$\begin{aligned}\mu_1 &= 1-\frac{(\alpha-1)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n},\\ \mu_2 &= \frac{(\alpha-1)(3+\alpha)^n\alpha^{n-1}}{(1+\alpha)^n(2+\alpha)^n},\\ \lambda &= 1-\frac{2}{3}\left[\left(\frac{1-2\alpha}{\alpha}\right)\left(\frac{\alpha^2+2\alpha}{\alpha^2+2\alpha+1}\right)^n+3\right].\end{aligned} \tag{3.32}$$

It is readily seen that for $\alpha \ge 1$, both μ_1 and μ_2 are nonnegative and $\mu_1+\mu_2 = 1$. Further, with a little bit of manipulation, it can be shown that $|\lambda| \le 1$ and the coefficients of z and z^2 in the expansion of h are respectively those given by (3.29) and (3.30). Hence $h \in P$ and thus $|a_4| \le 2\alpha^{n-1}/(3+\alpha)^n$, the second inequality in (3.3). This completes the proof of Theorem 3.1. □

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