

## FUNCTIONS STARLIKE WITH RESPECT TO OTHER POINTS

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ABSTRACT. In [7], Sakaguchi introduce the class of functions starlike with respect to symmetric points. We extend this class. For  $0 \leq \beta < 1$ , let  $S_S^*(\beta)$  be the class of normalised analytic functions  $f$  defined in the open unit disc  $D$  such that  $\operatorname{Re} zf'(z)/(f(z)-f(-z)) > \beta$ , for some  $z \in D$ . In this paper, we introduce 2 other similar classes  $S_C^*(\beta)$ ,  $S_{SC}^*(\beta)$  as well as give sharp results for the real part of some function for  $f \in S_S^*(\beta)$ ,  $S_C^*(\beta)$  and  $S_{SC}^*(\beta)$ . The behaviour of certain integral operators are also considered.

KEY WORDS AND PHRASES. *Starlike functions of order  $\beta$ , functions starlike with respect to symmetrical points, close-to-convex, integral operators.*

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### 1. INTRODUCTION.

Let  $S$  be the class of analytic functions  $f$ , univalent in the unit disc  $D = \{z : |z| < 1\}$ , with

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (1.1)$$

For  $0 \leq \beta < 1$ , denote by  $S^*(\beta)$ , the class of starlike functions of order  $\beta$ . Then  $f \in S^*(\beta)$  if, and only if, for  $z \in D$ ,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \beta.$$

In [7], Sakaguchi introduced the class  $S_S^*$  of analytic functions  $f$ , normalised by (1.1) which are starlike with respect to symmetrical points. We begin by defining the class  $S_S^*$ , which is contained in  $K$ , the class of close-to-convex functions.

#### DEFINITION 1.

A function  $f \in S_S^*$  if, and only if, for  $z \in D$ ,

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)-f(-z)} \right\} > 0.$$

We now extend this definition as follows:

DEFINITION 2.

A function  $f$  with normalisations (1.1) is said to be starlike of order  $\beta$ , with respect to symmetric points if, and only if, for  $z \in D$  and  $0 \leq \beta < 1$ ,

$$\operatorname{Re} \left( \frac{2zf'(z)}{f(z)-f(-z)} \right) > \beta.$$

We denote this class by  $S_S^*(\beta)$  and note that  $S_S^* = S_S^*(0)$ .

In the same manner, we define the following new classes of close-to-convex functions, which are generalisations of the classes in El-Ashwah and Thomas [2].

DEFINITION 3.

A function  $f$  normalised by (1.1) is said to be starlike of order  $\beta$ , with respect to conjugate points if, and only if, for  $z \in D$  and  $0 \leq \beta < 1$ ,

$$\operatorname{Re} \left( \frac{2zf'(z)}{f(z)+f(\bar{z})} \right) > \beta.$$

We denote this class by  $S_C^*(\beta)$ .

DEFINITION 4.

A function  $f$  normalised by (1.1) is said to be starlike of order  $\beta$ , with respect to symmetric conjugate points if, and only if, for  $z \in D$  and  $0 \leq \beta < 1$ ,

$$\operatorname{Re} \left( \frac{2zf'(z)}{f(z)-f(\bar{z})} \right) > \beta.$$

We denote this class by  $S_{sc}^*(\beta)$ .

REMARK.

The class  $S_S^*$  has been studied by several authors, (eg. Wu [9] and Stankiewicz [8]).

For  $f \in S_S^*(\beta)$ , Owa et al [4] proved that for  $\frac{1}{4} \leq \beta < \frac{1}{2}$ ,

$$\operatorname{Re} \left( \frac{f(z)-f(-z)}{z} \right) > \frac{2}{3-4\beta}, \quad z \in D.$$

## 2. RESULTS.

THEOREM 1.

Let  $f \in S_S^*(\beta)$ , then for  $z = re^{i\theta} \in D$ ,

$$\operatorname{Re} \left( \frac{f(z)-f(-z)}{z} \right)^{1/(1-\beta)} \geq \frac{2^{1/(1-\beta)}}{1+r^2} > 2^{\beta/(1-\beta)}.$$

The result is sharp for  $f_0$  given by

$$f_0(z) - f_0(-z) = 2z(1+z^2)^{\beta-1}.$$

To prove Theorem 1, we first require the following lemma.

LEMMA 1.

Let  $g \in S_S^*(\beta)$  and be odd. Then for  $z = re^{i\theta} \in D$ ,

$$\operatorname{Re} \left( \frac{g(z)}{z} \right)^{1/(1-\beta)} \geq \frac{1}{1+r^2}.$$

PROOF OF LEMMA.

Pinchuk [5] showed that if  $F \in S^*(\beta)$ , then for  $z = re^{i\theta} \in D$ ,

$$\left| \left( \frac{z}{F(z)} \right)^{1/2(1-\beta)} - 1 \right| \leq r. \tag{2.1}$$

Since  $g$  is odd, we may write  $[g(z)]^2 = F(z^2)$ , so that (2.1) gives

$$\left| \left( \frac{z}{g(z)} \right)^{1/(1-\beta)} - 1 \right| \leq r^2,$$

where on squaring both sides, gives

$$\left( \frac{r}{|g(z)|} \right)^{2/(1-\beta)} - 2 \operatorname{Re} \left[ \left( \frac{z}{g(z)} \right)^{1/(1-\beta)} \right] + 1 \leq r^4.$$

Thus

$$\begin{aligned} 2 \operatorname{Re} \left[ \left( \frac{g(z)}{z} \right)^{1/(1-\beta)} \right] &\geq (1-r^4) \left( \frac{|g(z)|}{r} \right)^{2/(1-\beta)} + 1 \\ &\geq \frac{1-r^4}{(1+r^2)^2} + 1, \end{aligned}$$

where we have used the inequality [6]

$$|g(z)| \geq \frac{r}{(1+r^2)^{1-\beta}},$$

for odd starlike functions of order  $\beta$ .

The Lemma now follows at once.

PROOF OF THEOREM 1.

Since  $f \in S^*_s(\beta)$ , it follows that we may write

$$g(z) = \frac{f(z) - f(-z)}{2},$$

for  $g$  an odd starlike function of order  $\beta$ . An application of Lemma 1 proves the Theorem.

Results analogous to Theorem 1 can also be found for the classes  $S^*_c(\beta)$  and  $S^*_{sc}(\beta)$ .

THEOREM 2.

Let  $f \in S^*_c(\beta)$ . Then for  $z = re^{i\theta} \in D$ ,

$$\operatorname{Re} \left( \frac{f(z) + \overline{f(\bar{z})}}{z} \right)^{1/2(1-\beta)} \geq \frac{(\sqrt{2})^{1/(1-\beta)}}{1+r} > 2^{(2\beta-1)/2(1-\beta)}.$$

The result is sharp for  $f(z) + \overline{f(\bar{z})} = 2z(1+z)^{2(\beta-1)}$

PROOF

Since  $f \in S^*_c(\beta)$ , it is easy to see that, if

$$F(z) = \frac{f(z) + \overline{f(\bar{z})}}{2}$$

then  $F \in S^*(\beta)$ . Using the same techniques as in the proof of Lemma 1, it follows from (2.1) that

$$\operatorname{Re} \left( \frac{F(z)}{z} \right)^{1/2(1-\beta)} \geq \frac{1}{1+r}.$$

The result now follows immediately.

Similarly, we have the following result, which we state without proof.

THEOREM 3.

Let  $f \in S_{sc}^*(\beta)$ . Then for  $z = re^{i\theta} \in D$ ,

$$\operatorname{Re} \left( \frac{f(z) - \overline{f(-\bar{z})}}{z} \right)^{1/2(1-\beta)} \geq \frac{2^{1/2(1-\beta)}}{1+r} > 2^{(2\beta-1)/2(1-\beta)}.$$

The result is sharp for  $f(z) - \overline{f(-\bar{z})} = 2z(1+z)^{2(\beta-1)}$

We now consider the results of some integral operators. In [1] Das and Singh, obtained analogous results of the Libera integral operator. They proved that for  $f \in S_S^*(0)$ , the function  $h$  given by

$$h(z) = \frac{1}{2} \int_0^z t^{-1} [f(t) - f(-t)] dt$$

also belongs to  $S_S^*(0)$ .

The result below generalises that of Das and Singh.

THEOREM 4.

Let  $f \in S_S^*(\beta)$ . Then the function  $H$  defined by

$$H(z) = \frac{a+1}{2z^a} \int_0^z t^{a-1} [f(t) - f(-t)] dt, \tag{2.2}$$

also belongs to  $S_S^*(\beta)$  for  $z \in D$  and  $a + \beta > 0$ .

We first require the following Lemma due to Miller and Mocanu [5].

LEMMA 2.

Let  $M$  and  $N$  be analytic in  $D$  with  $M(0) = N(0) = 0$  and let  $\beta$  be any real number. If  $N(z)$  maps  $D$  onto a (possibly many sheeted) region which is starlike with respect to the origin, then for  $z \in D$ ,

$$\operatorname{Re} \frac{M'(z)}{N'(z)} > \beta \implies \operatorname{Re} \frac{M(z)}{N(z)} > \beta,$$

and

$$\operatorname{Re} \frac{M'(z)}{N'(z)} < \beta \implies \operatorname{Re} \frac{M(z)}{N(z)} < \beta.$$

PROOF OF THEOREM 4.

(2.2) gives,

$$\begin{aligned} \frac{2zH'(z)}{H(z) - H(-z)} &= \frac{z^a [f(z) - f(-z)] - a \int_0^z t^{a-1} [f(t) - f(-t)] dt}{\int_0^z t^{a-1} [f(t) - f(-t)] dt} \\ &= \frac{M(z)}{N(z)}, \text{ say.} \end{aligned}$$

Note that  $M(0) = N(0) = 0$  and for  $f \in S_S^*(\beta)$ ,

$$\operatorname{Re} \left( 1 + \frac{zN''(z)}{N'(z)} \right) = a + \operatorname{Re} \left( \frac{z[f'(z)+f'(-z)]}{f(z)-f(-z)} \right) > a + \beta.$$

Thus  $N(z)$  is starlike if, and only if  $a > -\beta$ .

Furthermore, since

$$\operatorname{Re} \frac{M'(z)}{N'(z)} = \operatorname{Re} \left( \frac{z[f'(z)+f'(-z)]}{f(z)-f(-z)} \right) > \beta.$$

Lemma 2 shows that  $H \in S^*(\beta)$ .

Finally, we give the following analogous results for the classes  $S_c^*(\beta)$  and  $S_{sc}^*(\beta)$ .

THEOREM 5.

Let  $f \in S_c^*(\beta)$ . Then  $H$  defined by

$$H(z) = \frac{a+1}{2z^a} \int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt, \tag{2.3}$$

also belongs to  $S_c^*(\beta)$  for  $z \in D$  and  $a + \beta > 0$ .

PROOF.

Since  $f \in S_c^*(\beta)$ , (2.3) gives

$$\begin{aligned} \overline{\int_0^{\bar{z}} t^{a-1} [f(t) + \overline{f(\bar{t})}] dt} &= 2 \int_0^{\bar{z}} t^{a-1} \left( t + \sum_{n=2}^{\infty} \operatorname{Re} a_n t^n \right) dt \\ &= 2 \int_0^z t^{a-1} \left( t + \sum_{n=2}^{\infty} \operatorname{Re} a_n t^n \right) dt \\ &= \int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt. \end{aligned}$$

Thus

$$\begin{aligned} \frac{2z H'(z)}{H(z) + H(\bar{z})} &= \frac{z^a [f(z) + \overline{f(\bar{z})}] - a \int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt}{\int_0^z t^{a-1} [f(t) + \overline{f(\bar{t})}] dt} \\ &= \frac{M(z)}{N(z)}, \end{aligned}$$

where  $M(0) = N(0) = 0$  and  $N \in S^*$  for  $a + \beta > 0$ .

On using Lemma 2 it follows that  $H \in S_c^*(\beta)$ .

THEOREM 6.

Let  $f \in S_{sc}^*(\beta)$ . Then  $H$  defined by

$$H(z) = \frac{a+1}{2z^a} \int_0^z t^{a-1} [f(t) - \overline{f(-\bar{t})}] dt, \tag{2.4}$$

also belongs to  $S_{sc}^*(\beta)$  for  $z \in D$  and  $a + \beta > 0$ .

PROOF.

For  $f \in S_{sc}^*(\beta)$ , (2.4) gives

$$\begin{aligned} \overline{H(-\bar{z})} &= \frac{a+1}{(-z)^a} \int_0^{-\bar{z}} t^{a-1} [f(t) - \overline{f(-\bar{t})}] dt \\ &= \frac{a+1}{(-z)^a} \left\{ \frac{2(-z)^{a+1}}{a+1} + \sum_{n=2}^{\infty} \frac{(-z)^{n+a}}{n+a} (\overline{a_n} + (-1)^{n+1} a_n) \right\} \\ &= \frac{-(a+1)}{z^a} \int_0^z t^{a-1} [f(t) - \overline{f(-\bar{t})}] dt. \end{aligned}$$

As before, writing

$$\frac{2zH'(z)}{H(z) - H(-\bar{z})} = \frac{M(z)}{N(z)},$$

one can show that  $N \in S^*$  and hence using Lemma 2 the result follows.

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