

**PROBABILISTIC DERIVATION OF A BILINEAR SUMMATION FORMULA
 FOR THE MEIXNER-POLLACZEK POLYNOMIALS**

P. A. LEE

Department of Mathematics
 University of Malaya
 Kuala Lumpur 22-11
 Malaysia

(Received August 24, 1978)

ABSTRACT. Using the technique of canonical expansion in probability theory, a bilinear summation formula is derived for the special case of the Meixner-Pollaczek polynomials $\{\lambda_n^{(k)}(x)\}$ which are defined by the generating function

$$\sum_{n=0}^{\infty} \lambda_n^{(k)}(x) z^n / n! = (1+z)^{\frac{1}{2}(x-k)} / (1-z)^{\frac{1}{2}(x+k)}, \quad |z| < 1.$$

These polynomials satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} p_k(x) \lambda_m^{(k)}(ix) \lambda_n^{(k)}(ix) dx = (-1)^n n! \binom{k}{n} \delta_{m,n}, \quad i = \sqrt{-1}$$

with respect to the weight function

$$p_1(x) = \operatorname{sech} \pi x$$

$$p_k(x) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \operatorname{sech} \pi x_1 \operatorname{sech} \pi x_2 \dots$$

$$\operatorname{sech} \pi(x - x_1 - \dots - x_{k-1}) dx_1 dx_2 \dots dx_{k-1}, \quad k = 2, 3, \dots$$

KEY WORDS AND PHRASES. Meixner-Pollaczek polynomials, orthogonal polynomials, bilinear summation formula, bivariate distribution, canonical expansion, Runge identity, G-functions.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. Primary 42A60, Secondary 33A70 62H20.

1. INTRODUCTION

Let U be a Cauchy random variable with the probability density function (p.d.f.)

$$f(u) = \frac{1}{\pi} \frac{1}{1 + u^2}, \quad -\infty < u < \infty.$$

Consider the transformation $U = \sinh \pi V$. The p.d.f. of V is

$$p(v) = \operatorname{sech} \pi v, \quad -\infty < v < \infty. \quad (1)$$

This is the hyperbolic secant distribution considered by Baten [2], and is a special case of the generalized hyperbolic secant distribution treated by Harkness and Harkness [10].

Let X_1 and X_2 be two random variables having additive random elements in common [6], i.e.

$$X_1 = V_1 + V_2$$

$$X_2 = V_2 + V_3$$

where V_i ($i = 1, 2, 3$) are mutually independent random variables each having the p.d.f. given in (1). The joint p.d.f. $p(x_1, x_2)$ of X_1 and X_2 is easily shown to be

$$\begin{aligned} p(x_1, x_2) &= \int_{-\infty}^{\infty} \operatorname{sech} \pi z \operatorname{sech} \pi(x_1 - z) \operatorname{sech} \pi(x_2 - x_1 + z) dz \\ &= \frac{1}{2} \operatorname{sech} \frac{\pi x_1}{2} \operatorname{sech} \frac{\pi x_2}{2} \operatorname{sech} \frac{\pi(x_2 - x_1)}{2}, \quad -\infty < x_1 < \infty, \\ &\quad -\infty < x_2 < \infty. \end{aligned} \quad (2)$$

The marginal p.d.f.'s for X_1 and X_2 are respectively

$$\begin{aligned}
 g(x_1) &= \int_{-\infty}^{\infty} p(x_1, x_2) dx_2 = 2x_1 \operatorname{cosech} \pi x_1, \\
 h(x_2) &= \int_{-\infty}^{\infty} p(x_1, x_2) dx_1 = 2x_2 \operatorname{cosech} \pi x_2.
 \end{aligned}
 \tag{3}$$

The orthogonal polynomials with the above marginals as weight function are related to the Euler numbers and have been discussed by Carlitz in [4].

Specifically, for the weight function

$$\begin{aligned}
 p_k(x) &= \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \operatorname{sech} \pi x_1 \operatorname{sech} \pi x_2 \dots \operatorname{sech} \pi(x - x_1 - x_2 \\
 &\quad \dots - x_{k-1}) dx_1 dx_2 \dots dx_{k-1}, \quad k = 2, 3, \dots \tag{4} \\
 p_1(x) &= \operatorname{sech} \pi x,
 \end{aligned}$$

the polynomials $\left\{ \lambda_n^{(k)}(x) \right\}$ with generating function

$$\sum_{n=0}^{\infty} \lambda_n^{(k)}(x) z^n / n! = (1+z)^{\frac{1}{2}(x-k)} / (1-z)^{\frac{1}{2}(x+k)}, \quad |z| < 1 \tag{5}$$

are the orthogonal polynomials in the interval $(-\infty, \infty)$ and satisfy the orthogonality condition

$$\int_{-\infty}^{\infty} p_k(x) \lambda_m^{(k)}(ix) \lambda_n^{(k)}(ix) dx = (-1)^n n! \binom{k}{n} \delta_{m,n} \tag{6}$$

where $i = \sqrt{-1}$ and $\delta_{m,n}$ denotes the Kronecker delta.

The explicit form of the orthogonal polynomial is given by Carlitz [4] as

$$\begin{aligned}
 \lambda_n^{(k)}(x) &= \sum_{r=0}^n 2^r \binom{\frac{1}{2}(x-k)}{r} \binom{n+k-1}{n-r} \\
 &= (-1)^n \binom{k}{n} {}_2F_1[-n, \frac{1}{2}(x+k); k; 2].
 \end{aligned}
 \tag{7}$$

The last result follows easily from the following well-known generating function for the Gaussian hypergeometric function ${}_2F_1$ [7, p. 82]

$$(1+z)^{b-c} [1+(1-x)z]^{-b} = \sum_{n=0}^{\infty} \binom{-c}{n} {}_2F_1[-n, b; c; x] z^n$$

$$|z| < 1, \quad |z - zx| < 1.$$

A related system of polynomials has been discussed by Bateman [1] who referred to them as the Mittag-Leffler polynomials. It happens that both the polynomials discussed by Bateman and Carlitz are but special cases of the system of orthogonal polynomials first discussed by Meixner [11] and later independently by Pollaczek [12]. Following the notation of [8, p. 219] (See also [5, p. 184]), the Meixner-Pollaczek polynomials are given explicitly by

$$P_n^{(\alpha)}(x; \phi) = \frac{(2\alpha)_n}{n!} e^{in\phi} {}_2F_1[-n, \alpha + ix; 2\alpha; 1 - e^{-2i\phi}] \quad (8)$$

where $\alpha > 0$, $0 < \phi < \pi$ and $-\infty < x < \infty$.

These polynomials are orthogonal with respect to the weight function

$$\omega^{(\alpha)}(x; \phi) = \frac{(2 \sin \phi)^{2\alpha-1}}{\pi} e^{-(\pi-2\phi)x} |\Gamma(\alpha + ix)|^2.$$

The orthogonality relation is given by

$$\int_{-\infty}^{\infty} \omega^{(\alpha)}(x; \phi) P_m^{(\alpha)}(x; \phi) P_n^{(\alpha)}(x; \phi) dx = \frac{\Gamma(2\alpha + n)}{n!} \operatorname{cosec} \phi \delta_{m,n}.$$

A generating function for $P_n^{(\alpha)}(x; \phi)$ is

$$\sum_{n=0}^{\infty} t^n P_n^{(\alpha)}(x; \phi) = (1 - te^{i\phi})^{-\alpha+ix} (1 - te^{-i\phi})^{-\alpha-ix}, \quad |t| < 1. \quad (9)$$

It is clear when comparing (7) with (8) or (5) with (9) that

$$(-i)^n \lambda_n^{(k)}(ix) = n! P_n^{(k/2)}(x/2; \pi/2), \quad k = 1, 2, \dots; n = 0, 1, 2, \dots$$

and thus $\lambda_n^{(k)}(x)$ may be regarded as a special case of the Meixner-Pollaczek polynomials.

2. A BILINEAR SUMMATION FORMULA

From the generating function in (5) it is immediately clear that $\lambda_n^{(k)}(x)$ satisfies the following so-called Runge-type identity

$$\lambda_n^{(k_1+k_2)}(x_1 + x_2) = \sum_{r=0}^n \binom{n}{r} \lambda_r^{(k_1)}(x_1) \lambda_{n-r}^{(k_2)}(x_2), \quad k_1, k_2 = 1, 2, 3, \dots \quad (10)$$

and all n .

It has been shown that the result in (10) is both necessary and sufficient for the joint p.d.f. in (2) to possess a bilinear expansion (also called a canonical expansion in statistical literature) of the form [6]

$$p(x_1, x_2) = g(x_1)h(x_2) \sum_{r=0}^{\infty} \rho_n \theta_n(x_1) \phi_n(x_2)$$

where the canonical variables $\{\theta_n(x)\}$ ($\{\phi_n(x)\}$) are a complete set of orthonormal polynomials with weight function $g(x)$ ($h(x)$). The canonical correlation is

$$\rho_n = E[\theta_n(X_1)\phi_n(X_2)]$$

where E denotes the expectation operation.

For the joint p.d.f. in (2) with the equal marginal p.d.f.'s given in (3), we note that the canonical variable in this case is

$$\theta_n(x) = \phi_n(x) = \frac{i^{-n}}{\sqrt{n!(2)_n}} \lambda_n^{(2)}(ix).$$

The canonical correlation is

$$\begin{aligned} \rho_n &= E[\theta_n(X_1)\phi_n(X_2)] \\ &= \frac{(-1)^{-n}}{n!(2)_n} E\{\lambda_n^{(2)}[i(V_1 + V_2)]\lambda_n^{(2)}[i(V_2 + V_3)]\} \\ &= \frac{(-1)^{-n}}{n!(2)_n} \sum_{s=0}^n \sum_{r=0}^n \binom{n}{s} \binom{n}{r} E[\lambda_r^{(1)}(iV_1)]. \end{aligned}$$

$$\begin{aligned}
 & \cdot E[\lambda_{n-r}^{(1)}(iv_2)\lambda_s^{(1)}(iv_2)]E[\lambda_{n-s}^{(1)}(iv_3)] \\
 &= \frac{(-1)^{-n}}{n!(2)_n} E[\lambda_n^{(1)}(iv_2)]^2 \\
 &= \frac{1}{n+1} \cdot \tag{11}
 \end{aligned}$$

We therefore have the following interesting bilinear summation formula for the Meixner-Pollaczek polynomials

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{(-1)^n}{[(n+1)!]^2} \lambda_n^{(2)}(ix_1)\lambda_n^{(2)}(ix_2) &= \sinh \frac{\pi x_1}{2} \sinh \frac{\pi x_2}{2} \operatorname{sech} \frac{\pi(x_2 - x_1)}{2} / (2x_1 x_2) \\
 & \quad -\infty < x_1 < \infty \quad \text{and} \quad -\infty < x_2 < \infty. \tag{12}
 \end{aligned}$$

3. A GENERALIZATION

Consider the following more general scheme of additive random variables as in [9].

Let $\{\xi_i\}$ for $i = 1, 2, \dots, n - m$, $\{\eta_i\}$ for $i = 1, 2, \dots, m$ and $\{\zeta_i\}$ for $i = 1, 2, \dots, n_2 - m$ where $1 \leq m < \min(n_1, n_2)$ be $(n_1 + n_2 - m)$ mutually independent random variables each having the p.d.f. given in (1).

Define

$$U = \sum_{i=1}^{n_1-m} \xi_i, \quad V = \sum_{i=1}^m \eta_i, \quad W = \sum_{i=1}^{n_2-m} \zeta_i$$

$$X_1 = U + V$$

$$X_2 = V + W.$$

It is clear that the joint characteristic function $\phi(\omega_1, \omega_2)$ of X_1 and X_2 is

$$\phi(\omega_1, \omega_2) = E[\exp(i\omega_1 X_1 + i\omega_2 X_2)]$$

$$\begin{aligned}
 &= E\{\exp[i\omega_1 U + i\omega_2 W + i(\omega_1 + \omega_2)V]\} \\
 &= \operatorname{sech}^{n_1 - m} \left(\frac{\omega_1}{2}\right) \operatorname{sech}^{n_2 - m} \left(\frac{\omega_2}{2}\right) \operatorname{sech}^m \left(\frac{\omega_1 + \omega_2}{2}\right)
 \end{aligned}$$

since

$$\begin{aligned}
 E \left[e^{i\omega \xi_i} \right] &= E \left[e^{i\omega \eta_j} \right] = E \left[e^{i\omega \xi_k} \right] \\
 &= \int_{-\infty}^{\infty} e^{i\omega v} \operatorname{sech} \pi v \, dv \\
 &= \operatorname{sech} \left(\frac{\omega}{2}\right) \quad \text{for } 1 \leq i \leq n_1 - m, \\
 &\quad 1 \leq j \leq m, \\
 &\quad 1 \leq k \leq n_2 - m.
 \end{aligned}$$

The joint p.d.f. in question is therefore

$$\begin{aligned}
 p(x_1, x_2) &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\omega_1 x_1 - i\omega_2 x_2} \operatorname{sech}^{n_1 - m} \left(\frac{\omega_1}{2}\right) \operatorname{sech}^{n_2 - m} \left(\frac{\omega_2}{2}\right) \\
 &\quad \operatorname{sech}^m \left(\frac{\omega_1 + \omega_2}{2}\right) d\omega_1 d\omega_2 \tag{13}
 \end{aligned}$$

and the marginal p.d.f.'s for X_1 and X_2 are respectively

$$\begin{aligned}
 g(x_1) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_1 x_1} \operatorname{sech}^{n_1} \left(\frac{\omega_1}{2}\right) d\omega_1 \\
 &= \frac{1}{\pi} \frac{2^{n_1 - 1}}{(n_1 - 1)!} \left| \Gamma \left(\frac{n_1}{2} + ix_1\right) \right|^2 \\
 h(x_2) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_2 x_2} \operatorname{sech}^{n_2} \left(\frac{\omega_2}{2}\right) d\omega_2 \\
 &= \frac{1}{\pi} \frac{2^{n_2 - 1}}{(n_2 - 1)!} \left| \Gamma \left(\frac{n_2}{2} + ix_2\right) \right|^2
 \end{aligned}$$

on using the fact that [3, p. 31]

$$\int_{-\infty}^{\infty} e^{-ivx} [\operatorname{sech}(\beta x + \gamma)]^{\mu+1} dx = \frac{2^\mu}{\beta} \frac{\left| \Gamma\left(\frac{1 + \mu + iv/\beta}{2}\right) \right|^2}{\Gamma(\mu + 1)} e^{iv\gamma/\beta}. \quad (14)$$

The respective canonical variables are

$$\theta_n(x_1) = \frac{i^{-n}}{\sqrt{n!(n_1)_n}} \lambda_n^{(n_1)}(ix_1)$$

$$\phi_n(x_2) = \frac{i^{-n}}{\sqrt{n!(n_2)_n}} \lambda_n^{(n_2)}(ix_2).$$

By a repeated application of the Runge-type identity in (10) analogous to the derivation leading to the result in (11), it may be shown that the canonical correlation in this case is

$$\rho_n = \frac{\binom{m}{n}}{\sqrt{\binom{n_1}{n} \binom{n_2}{n}}}, \quad 1 \leq m < \min(n_1, n_2).$$

On the other hand, note that from (14)

$$\begin{aligned} & \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega_1 x_1 - i\omega_2 x_2} \operatorname{sech}^m\left(\frac{\omega_1 + \omega_2}{2}\right) d\omega_1 d\omega_2 \\ &= \frac{2^{m-1}}{\pi(m-1)!} \left| \Gamma\left(\frac{m}{2} + ix_1\right) \right|^2 \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-i\omega_2(x_2 - x_1)] d\omega_2 \\ &= \frac{2^{m-1}}{\pi(m-1)!} \left| \Gamma\left(\frac{m}{2} + ix_1\right) \right|^2 \delta(x_2 - x_1) \end{aligned}$$

where $\delta(x)$ denotes the Dirac delta function.

A double convolution operation applied to (13) then yields the following expression for $p(x_1, x_2)$

$$p(x_1, x_2) = \frac{2^{n_1+n_2+m-3}}{\pi^3(m-1)!(n_1-m-1)!(n_2-m-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{n_1-m}{2} + iu\right) \right|^2 \cdot \left| \Gamma\left(\frac{n_2-m}{2} + iv\right) \right|^2 \left| \Gamma\left(\frac{m}{2} + i(x_1-u)\right) \right| \delta(x_2 - v - x_1 + u) dudv$$

$$= \frac{2^{n_1+n_2+m-3}}{\pi^3 (m-1)! (n_1-m-1)! (n_2-m-1)!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{n_1-m}{2} + iu\right) \right|^2 \cdot \left| \Gamma\left(\frac{m}{2} + i(x_1-u)\right) \right|^2 \left| \Gamma\left(\frac{n_2-m}{2} + i(x_2-x_1+u)\right) \right|^2 du. \tag{15}$$

Finally, the result in (15) may be rewritten into the following Barnes type contour integral

$$p(x_1, x_2) = \frac{2^{n_1+n_2+m-2}}{\pi^2 (m-1)! (n_1-m-1)! (n_2-m-1)!} \cdot \frac{1}{2\pi i} \int_{-i\infty}^{-i\infty} \Gamma\left(\frac{n_1-m}{2} + s\right) \Gamma\left(\frac{m}{2} - ix_1 + s\right) \Gamma\left(\frac{n_2-m}{2} + i(x_2-x_1) + s\right) \Gamma\left(\frac{n_1-m}{2} - s\right) \Gamma\left(\frac{m}{2} + ix_1 - s\right) \Gamma\left(\frac{n_2-m}{2} - i(x_2-x_1) - s\right) ds$$

which may be evaluated in terms of a sum of ${}_3F_2$ functions [13, p. 133] or, perhaps more conveniently, in terms of Meijer's G-function as follows [7, Sec. 5.3]

$$p(x_1, x_2) = \frac{2^{n_1+n_2-m-2}}{\pi^2 (m-1)! (n_1-m-1)! (n_2-m-1)!} \cdot G_{3,3}^{3,3} \left[\begin{matrix} 1 - \frac{n_1-m}{2}, & 1 - \frac{m}{2} + ix_1, & 1 - \frac{n_2-m}{2} - i(x_2-x_1) \\ \frac{n_1-m}{2}, & \frac{m}{2} + ix_1, & \frac{n_2-m}{2} - i(x_2-x_1) \end{matrix} \right].$$

The existence of a diagonal expansion then implies the following summation formula

$$\sum_{n=0}^{\infty} \frac{(-1)^n (m)_n}{(n_1)_n (n_2)_n n!} \lambda_n(x_1) \lambda_n(x_2) = \frac{(n_1-1)! (n_2-1)!}{2^m (m-1)! (n_1-m-1)! (n_2-m-1)!} \cdot \frac{1}{\Gamma\left(\frac{n_1}{2} + x_1\right) \Gamma\left(\frac{n_1}{2} - x_1\right) \Gamma\left(\frac{n_2}{2} + x_2\right) \Gamma\left(\frac{n_2}{2} - x_2\right)}.$$

$$G_{3,3}^{3,3} \left(\begin{matrix} 1 - \frac{n_1 - m}{2}, & 1 - \frac{m}{2} + x_1, & 1 - \frac{n_2 - m}{2} - (x_2 - x_1) \\ \frac{n_1 - m}{2}, & \frac{m}{2} + x_1, & \frac{n_2 - m}{2} - (x_2 - x_1) \end{matrix} \right) \quad (16)$$

for $1 \leq m < \min(n_1, n_2)$, $-\infty < x_1 < \infty$, $-\infty < x_2 < \infty$.

It is perhaps interesting to note in passing that a comparison of the two results in (12) and (16) allows us to deduce the following special case of the G-function, viz.

$$G_{3,3}^{3,3} \left(\begin{matrix} \frac{1}{2}, & \frac{1}{2} + x_1, & \frac{1}{2} + x_1 - x_2 \\ \frac{1}{2}, & \frac{1}{2} + x_1, & \frac{1}{2} + x_1 - x_2 \end{matrix} \right) = \frac{\pi^2}{4} \sec \frac{\pi x_1}{2} \sec \frac{\pi x_2}{2} \sec \frac{\pi(x_2 - x_1)}{2}.$$

ACKNOWLEDGEMENT. The author is grateful to the referee for his helpful comments, in particular regarding the connection of the system of orthogonal polynomials discussed by Carlitz with that of the Meixner-Pollaczek polynomials.

REFERENCES

1. Bateman, H. The Polynomials of Mittag-Leffler, Proc. Nat. Acad. Sci. (U.S.A.) **26** (1940) 491-496.
2. Baten, W.D. The Probability Law for the Sum of n Independent Variables, each subject to the Law $(1/(2h)) \operatorname{sech}(\pi x/(2h))$, Bull. Amer. Math. Soc. **40** (1934) 284-290.
3. Buchholz, H. The Confluent Hypergeometric Functions, Springer Verlag, New York, 1969.
4. Carlitz, L. Bernoulli and Euler Numbers and Orthogonal Polynomials, Duke Math. J. **26** (1959) 1-15.
5. Chihara, T.S. An Introduction to Orthogonal Polynomials, Gordon and Breach, New York, 1978.
6. Eagleson, G.K. and H.O. Lancaster. The Regression System of Sums with Random Elements in Common, Aust. J. Statist. **9** (1967) 119-125.
7. Erdélyi, A., A. Magnus, F. Oberhettinger and F. Tricomi. Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York, 1953.
8. Erdélyi, A., A. Magnus, F. Oberhettinger and F. Tricomi. Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953.

9. Hamden, M.A. Canonical Expansion of the Bivariate Binomial Distribution with Unequal Marginal Indices, Int. Statist. Rev. 40 (1972) 277-280.
10. Harkness, W.L. and M.L. Harkness. Generalized Hyperbolic Secant Distributions, J. Amer. Statist. Assoc. 63 (1968) 329-337.
11. Meixner, J. Orthogonale Polynomsysteme mit einem besonderen Gestalt der erzeugenden Funktion, J. London Math. Soc. 9 (1934) 6-13.
12. Pollaczek, F. Sur une famille de polynômes orthogonaux qui contient les polynômes d'Hermite et de Laguerre comme cas limites, C.R. Acad. Sci., Paris 230 (1950) 1563-1565.
13. Slater, L.J. Generalized Hypergeometric Functions, Cambridge University Press, London, 1966.



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

