\((\eta, \eta_\alpha, \theta)\)-Einstein real hypersurfaces in complex two-plane Grassmannians

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Abstract. In this paper, we introduce the notion of \((\eta, \eta_\alpha, \theta)\)-Einstein real hypersurfaces in complex two-plane Grassmannians. We show that there does not exist any \((\eta, \eta_\alpha, \theta)\)-Einstein real hypersurface in complex two-plane Grassmannians such that \(\xi\) is tangent to \(\mathcal{D}\). Some examples of \((\eta_\alpha, \theta)\)-Einstein real hypersurfaces are given.

1 Introduction

A Riemannian manifold is said to be Einstein if the Ricci tensor \(S\) is given by \(S = \rho \mathcal{I}\), where \(\rho\) is a constant. The Einstein condition can be generalized in a natural manner for those spaces with certain additional geometric structures.

An almost contact metric manifold \((M, \phi, \eta, \xi, g)\) is said to be \(\eta\)-Einstein if it satisfies \(S = f_1 \mathcal{I} + f_2 \xi \otimes \eta\), for some functions \(f_1, f_2\) on \(M\). Similar notion was also introduced in almost 3-contact metric geometry. Suppose now \(M\) is a manifold with an almost 3-contact metric structure \((\phi_\alpha, \eta_\alpha, \xi_\alpha, g), \alpha \in \{1, 2, 3\}\). If the Ricci tensor \(S\) satisfies \(S = f_1 \mathcal{I} + f_2 \sum_{\alpha=1}^{3} \xi_\alpha \otimes \eta_\alpha\), where \(f_1, f_2\) are functions on \(M\), then \(M\) is said to be \(\eta_\alpha\)-Einstein.

For a Kähler or quaternionic Kähler manifold, its real hypersurfaces (i.e., submanifolds of real codimension one) naturally inherited an almost contact metric (resp. almost 3-contact metric) structure from the Kähler (resp. quaternionic Kähler) structure of the ambient manifold. The study of real hypersurfaces in a

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Kähler (resp. quaternionic Kähler) manifold has become a branch of almost contact metric (resp. almost 3-contact metric) geometry.

In the case of non-flat complex space form, \(\eta\)-Einstein real hypersurfaces were classified in [3, 5, 9]. In the classification, we obtain that \(f_1, f_2\) must be constant and there does not exist any Einstein real hypersurface in a non-flat complex space form.

On the other hand, \(\eta_a\)-Einstein real hypersurfaces in non-flat quaternionic space forms were studied in [4, 8, 10], and a complete classification of such spaces could be deduced from a result in [10]. According to their results, we see that \(f_1, f_2\) must also be constant as in the Kählerian case. Moreover, only quaternionic projective spaces \(\mathbb{H}P^m\) admit an Einstein real hypersurface, which must be a tube of radius \(r \in [0, \pi/2]\) over a totally geodesic \(\mathbb{H}P^{m-1}\) with \(\cot^2 r = 2m\).

**Remark 1.1.** \(\eta\)-Einstein and \(\eta_a\)-Einstein real hypersurfaces were studied under the name of pseudo-Einstein real hypersurfaces in the above mentioned papers. However, we shall not follow that terminology in this paper to avoid the confusion.

A complex two-plane Grassmannian \(G_2(C^{m+2})\) has some remarkable properties and structures. The most notable one being the fact that it is the unique compact irreducible Riemannian symmetric space with both a Kähler structure \(J\) and a quaternionic Kähler structure \(\mathcal{J}\) (cf. [1]). These geometric structures induce an almost contact 3-structure \((\phi_a, \xi_a, \eta_a), a \in \{1, 2, 3\}\) as well as an almost contact structure \((\phi, \xi, \eta)\) on its real hypersurfaces \(M\). These allow us to study both \(\eta\)-real hypersurfaces and \(\eta_a\)-real hypersurfaces in \(G_2(C^{m+2})\).

In this paper, we introduce a “generalized Einstein” condition on real hypersurface \(M\) in \(G_2(C^{m+2})\), apart from the impact due to both the almost contact and almost 3-contact structures on \(M\), which also characterizes the interaction between these two structures. A real hypersurface \(M\) in \(G_2(C^{m+2})\) is said to be \((\eta, \eta_a, \theta)\)-Einstein if it satisfies

\[
S = f_1 \mathbb{I} + f_2 \xi \otimes \eta + f_3 \sum_{a=1}^{3} \xi_a \otimes \eta_a + f_4 \theta.
\]

where \(f_1, f_2, f_3, f_4\), called the coefficient functions, are functions on \(M\) and \(\theta\) is a symmetric \((1,1)\)-tensor field on \(M\) given by \(\theta := \sum_{a=1}^{3} \eta_a(\xi)(\phi \phi_a - \xi \otimes \eta_a)\). For some special cases, we say that the real hypersurface \(M\) is \((\eta, \eta_a)\)-Einstein if \(f_4 = 0\); \((\eta_a, \theta)\)-Einstein if \(f_2 = 0\); etc.

In this paper, we shall first prove that there does not exist any \((\eta, \eta_a, \theta)\)-Einstein real hypersurface \(M\) in \(G_2(C^{m+2})\) with constant coefficient functions and \(\xi \in \mathcal{D}\), where \(\mathcal{D}^\perp := \{\xi_1, \xi_2, \xi_3\}\) (cf. Theorem 3.4). Next we show that real hypersurfaces of type \(A\) in \(G_2(C^{m+2})\) are \((\eta_a, \theta)\)-Einstein (cf. Theorem 4.1). With this result, we also obtain example of \(\eta_a\)-Einstein real hypersurfaces in \(G_2(C^{m+2})\).

**Remark 1.2.** \((\eta, \eta_a)\)-Einstein real hypersurfaces in \(G_2(C^{m+2})\) were considered in [11, 12].
2 Real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section, we summarize and list out some important formulae as well as well-known results in the theory of real hypersurfaces in complex two-plane Grassmannians (see [2, 7, 12] for details).

Denote the set of all complex 2-dimensional linear subspaces by $G_2(\mathbb{C}^{m+2})$ with Kähler structure $J$ and quaternionic Kähler structure $\mathcal{J}$. Let $M$ be a connected, oriented real hypersurface isometrically immersed in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, and $N$ a unit normal vector field on $M$. Denote by $g$ the Riemannian metric on $M$. A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathcal{J}$ on $G_2(\mathbb{C}^{m+2})$ induces a local almost contact 3-structure $(\phi_a, \xi_a, \eta_a, g)$ on $M$ by

\[ J_aX = \phi_aX + \eta_a(X)N, \quad J_aN = -\xi_a, \quad \eta_a(X) = g(X, \xi_a), \]

for any $X \in TM$. It follows that

\[ \phi_a\phi_{a+1} - \xi_a \otimes \eta_{a+1} = \phi_{a+2} = -\phi_{a+1}\phi_a + \xi_{a+1} \otimes \eta_a, \]
\[ \phi_a\xi_{a+1} = \xi_{a+2} = -\phi_{a+1}\xi_a \]

for $a \in \{1, 2, 3\}$. The indices in the preceding equations is taken modulo three.

Let $(\phi, \xi, \eta, g)$ be the almost contact metric structure on $M$ induced by $J$, i.e.,

\[ JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = g(X, \xi). \]

The two structures $(\phi, \xi, \eta, g)$ and $(\phi_a, \xi_a, \eta_a, g)$ are related as follows

\[ \phi_a\phi - \xi_a \otimes \eta = \phi\phi_a - \xi \otimes \eta_a; \quad \phi\xi_a = \phi_a\xi. \]

Next, we denote by $\nabla$ the Levi-Civita connection and $A$ the shape operator on $M$. Then

\[
(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi; \quad \nabla_X\xi = \phi AX \\
(\nabla_X \phi_a)Y = \eta_a(Y)AX - g(AX, Y)\xi_a + q_{a+2}(X)\phi_{a+1}Y - q_{a+1}(X)\phi_{a+2}Y \\
\nabla_X\xi_a = \phi_a AX + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2} \\
X\eta(\xi_a) = 2\eta_a(\phi AX) + \eta_{a+1}(\xi)q_{a+2}(X) - \eta_{a+2}(\xi)q_{a+1}(X)
\]

for any $X, Y \in TM$, where $q_a$ is a 1-form on $M$. We define a local symmetric $(1, 1)$-tensor field $\theta_a$ on $M$ by

\[ \theta_a := \phi_a\phi - \xi_a \otimes \eta. \]

Then we have the following identities

\[
(2.2) \begin{cases}
\text{tr} \theta_a = \eta(\xi_a), & \theta_a^2 - \phi\xi_a \otimes \eta_a\phi = I \\
\theta_a\xi = -\xi_a, & \theta_a\xi_a = -\xi, \quad \theta_a\phi\xi_a = \eta(\xi_a)\phi\xi_a \\
\theta_a\xi_{a+1} = \phi\xi_{a+2} = -\theta_{a+1}\xi_a \\
-\theta_a\phi\xi_{a+1} + \eta(\xi_{a+1})\phi\xi_a = \xi_{a+2} = \theta_{a+1}\phi\xi_a - \eta(\xi_a)\phi\xi_{a+1}.
\end{cases}
\]
Further, we can easily derive from (2.1) that

\[
(2.3) \quad (\nabla_X \phi_a)Y = (\nabla_X \phi_a)Y + \phi_a(\nabla_X \phi)Y - g(\nabla_X \xi, Y)\xi_a - \eta(Y)\nabla_X \xi_a = \eta_a(\phi Y)AX - g(AX, Y)\phi \xi_a + g_{a+2}(X)\theta_{a+1}Y - g_{a+1}(X)\theta_{a+2}Y
\]

\[
(2.4) \quad \nabla_X \phi \xi_a = \theta_a AX + \eta_a(\xi)AX + g_{a+2}(X)\phi \xi_{a+1} - g_{a+1}(X)\phi \xi_{a+2}.
\]

For each \(x \in M\), we define a subspace \(\mathcal{H}^\perp\) of \(T_xM\) by

\[
\mathcal{H}^\perp := \text{span}\{\xi, \xi_1, \xi_2, \phi_1, \phi_2, \phi_3\}.
\]

Let \(\mathcal{H}\) be the orthogonal complement of \(\mathcal{H}^\perp\) in \(T_xM\). Then \(\dim \mathcal{H} = 4m - 4\) (resp. \(\dim \mathcal{H} = 4m - 8\)) when \(\xi \in \mathcal{D}^\perp\) (resp. \(\xi \notin \mathcal{D}^\perp\)) and \(\mathcal{H}\) is invariant under \(\phi, \phi_a\) and \(\theta_a\). Moreover, \(\theta_{a\mid \mathcal{K}}\) has two eigenvalues: 1 and \(-1\). Denote by \(\mathcal{H}_a(\varepsilon)\) the eigenspace corresponds to the eigenvalue \(\varepsilon\) of \(\theta_{a\mid \mathcal{K}}\). Then \(\dim \mathcal{H}_a(1) = \dim \mathcal{H}_a(-1)\) is even, and

\[
\phi \mathcal{H}_a(\varepsilon) = \mathcal{H}_a(\varepsilon), \quad \phi_a \mathcal{H}_a(\varepsilon) = \mathcal{H}_a(\varepsilon), \quad \theta_{a\mid \mathcal{K}}(\mathcal{H}_a(\varepsilon)) = \mathcal{H}_a(-\varepsilon), \quad (a \neq b).
\]

Now we define \(\theta := \sum_{a=1}^3 \eta_a(\xi)\theta_a, \xi^\perp := \sum_{a=1}^3 \eta_a(\xi)\xi_a\) and \(\eta^\perp := \sum_{a=1}^3 \eta_a(\xi)\eta_a\).

Then by (2.2) and (2.3), we have

\[
(2.5) \quad \text{tr} \theta = \sum_{a=1}^3 \eta_a(\xi)^2 = ||\xi^\perp||^2
\]

\[
(2.6) \quad (\nabla_X \theta)Y = \sum_{a=1}^3 \{\theta_a X - 2g(AX, Y)\phi \xi_a\}
\]

It follows from (2.5) that the tensor field \(\theta\) provides an index to measure \(\xi\) for being tangential to \(\mathcal{D}\) or \(\mathcal{D}^\perp\).

**Lemma 2.1.** Let \(M\) be a real hypersurface in \(G_2(C^{m+2})\). Then \(0 \leq \text{tr} \theta \leq 1\).

Moreover, we have

(a) \(\text{tr} \theta = 0\) if and only if \(\xi \in \mathcal{D}\); and

(a) \(\text{tr} \theta = 1\) if and only if \(\xi \in \mathcal{D}^\perp\).
The equations of Gauss and Codazzi are given by

\[ R(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY \]
\[ + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \]
\[ + \sum_{a=1}^{3} \{ g(\phi_a Y,Z)\phi_a X - g(\phi_a X,Z)\phi_a Y - 2g(\phi_a X,Y)\phi_a Z \} \]
\[ + g(\theta a Y,Z)\theta a X - g(\theta a X,Z)\theta a Y \} \]

\[ (\nabla X A)Y - (\nabla Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \]
\[ + \sum_{a=1}^{3} (\eta_a(X)\phi_a Y - \eta_a(Y)\phi_a X - 2g(\phi_a X,Y)\xi_a \]
\[ + \eta_a(\phi X)\theta_a Y - \eta_a(\phi Y)\theta_a X \].

By the Gauss equation, the Ricci tensor \( S \) is given by

\[ (2.7) \quad S = hA - A^2 + (4m + 7)I + \theta - 3\xi \otimes \eta - \sum_{a=1}^{3} (3\xi_a \otimes \eta_a + \phi_a \otimes \eta_a) \],

where \( h := \text{tr} A \) is the mean curvature of \( M \).

Finally we state some well-known results.

**Lemma 2.2** ([7]). Let \( M \) be a real hypersurface in \( G_2(C^{m+2}) \), \( m \geq 3 \). If \( \xi \) is tangent to \( \mathcal{D} \), then \( A\phi \xi_a = 0 \), for \( a \in \{1, 2, 3\} \).

**Theorem 2.3** ([2]). Let \( M \) be a connected real hypersurface in \( G_2(C^{m+2}) \), \( m \geq 3 \). Then both \( \xi \) and \( \mathcal{D}^\perp \) are invariant under the shape operator of \( M \) if and only if

(A) \( M \) is an open part of a tube around a totally geodesic \( G_2(C^{m+1}) \) of \( G_2(C^{m+2}) \),

or

(B) \( m \) is even, say \( m = 2n \), and \( M \) is an open part of a tube around a totally geodesic \( \mathbb{H}P^m \) in \( G_2(C^{m+2}) \).

We say that a real hypersurface \( M \) in \( G_2(C^{m+2}) \) is of type \( A \) if it satisfies the first property in the characterization theorem given above. On the other hand, \( M \) is said to be of type \( B \) if it satisfies all properties in part (B). A connected orientable real hypersurface \( M \) in \( G_2(C^{m+2}) \) is said to be Hopf if the Reeb vector field \( \xi \) is invariant under the shape operator of \( M \). The following theorem provides the sufficient conditions of being a real hypersurface of type \( B \).
Theorem 2.4 ([6]). Let M be a connected orientable Hopf hypersurface in \( G_2(\mathbb{C}^{m+2}) \), \( m \geq 3 \). Then the Reeb vector \( \xi \) belongs to the distribution \( \mathcal{D} \) if and only if M is locally congruent to an open part of a real hypersurface of type \( B \).

3 \((\eta, \eta_0, \theta)\)-Einstein real hypersurfaces

We shall show that there does not exist any \((\eta, \eta_0, \theta)\)-real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) such that \( \xi \) is tangent to \( \mathcal{D} \) everywhere in this section. We begin with deriving a basic formula for such spaces.

Lemma 3.1. Let M be a \((\eta, \eta_0, \theta)\)-Einstein in \( G_2(\mathbb{C}^{m+2}) \) with constant coefficient functions, i.e.,

\[
S = f_1 + f_2 \xi \otimes \eta + f_3 \sum_{a=1}^{3} \xi_a \otimes \eta_a + f_4 \theta.
\]

where \( f_1, f_2, f_3, f_4 \) are constants. Then we have

(a) \( \text{grad tr} \, S = -4f_4 A \phi \xi^\perp \);

(b) \( f_3 \phi A \xi + f_3 \sum_{a=1}^{3} \phi_a A \xi_a + f_4 (A - h I) \phi \xi^\perp - f_4 \sum_{a=1}^{3} \theta_a A \phi \xi_a = 0 \).

Proof. By (2.5), the scalar curvature \( \text{tr} \, S \) has the form of

\[
\text{tr} \, S = (4m - 1) f_1 + f_2 + 3f_3 + f_4 ||\xi^\perp||^2.
\]

It follows from (2.1) that

\[
X \text{tr} \, S = 2 f_4 \sum_{a=1}^{3} \eta(\xi_a) X \eta(\xi_a) - 4f_4 \sum_{a=1}^{3} \eta(\xi_a) g(A \phi \xi_a, X) = -4f_4 g(A \phi \xi^\perp, X).
\]

Hence we obtain Statement (a). On the other hand, by using (2.1) and (2.6), we compute

\[
(\nabla_X S)Y = f_2 (g(\nabla_X \xi, Y) \xi + \eta(Y) \nabla_X \xi) + f_3 \sum_{a=1}^{3} (g(\nabla_X \xi_a, Y) \xi_a + \eta_0(Y) \nabla_X \xi_a)
\]

\[
+ f_4 (\nabla_X \theta) Y
\]

\[
= f_2 (g(\phi AX, Y) \xi + \eta(Y) \phi AX) + f_3 \sum_{a=1}^{3} (g(\phi_a AX, Y) \xi_a + \eta_0(Y) \phi_a AX)
\]

\[
+ f_4 (\eta^\perp(\phi Y) AX - g(AX, Y) \phi \xi^\perp) - 2f_4 \sum_{a=1}^{3} g(A \phi \xi_a, X) \theta_a Y.
\]
Hence, by the above equation and the Schur Lemma: \(2 \text{div } S = \text{grad tr } S\), we obtain Statement (b).

The following lemma can be obtained with the same arguments as in the proof of [3, Prop. 5.2]. We shall state without proof.

**Lemma 3.2.** Let \((M, \phi, \eta, \xi, g)\) be an almost contact metric manifold. Suppose there exist a symmetric \((1,1)\)-tensor field \(F\) on \(M\), a distribution \(\mathfrak{T}\) on \(M\) with \(\dim \mathfrak{T} \geq 4\), and two functions \(\lambda, \mu\) on \(M\) with \(\lambda < \mu\) such that

(a) \(\xi \in \mathfrak{T}\) everywhere;
(b) \(\mathfrak{T}\) is invariant under both \(F\) and \(\phi\);
(c) there is an orthogonal decomposition \(\mathfrak{T} = \mathfrak{T}_\lambda \oplus \mathfrak{T}_\mu\) such that \(FX = \lambda X\) (resp. \(FY = \mu Y\)) for any \(X \in \mathfrak{T}_\lambda\) (resp. \(Y \in \mathfrak{T}_\mu\));
(d) \((\nabla_X F)Y - (\nabla_Y F)X = \epsilon(\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \omega(X, Y))\), for any \(X, Y \in \mathfrak{T}\), where \(\epsilon\) is a nonvanishing function and \(\omega\) is a \((1,2)\)-tensor field on \(M\) such that \(\omega(X, Y) \perp \mathfrak{T}\) for any \(X, Y \in \mathfrak{T}\).

Then \(\xi\) is tangent to either \(\mathfrak{T}_\lambda\) or \(\mathfrak{T}_\mu\) everywhere. In other words, \(\xi\) is an eigenvector for \(F\).

Next, we give an elementary algebraic lemma.

**Lemma 3.3.** Let \(F\) be a symmetric endomorphism of a finite dimensional inner product space \(V\) and \(X, Y \in V\) with \(X \perp Y\). Suppose \(PX = \sigma X\) and \(PY = \tau Y\), where \(P = F^2 - hF\) and \(h, \sigma, \tau\) are scalars. If \(\sigma \neq \tau\), then \(FX \perp Y\).

**Proof.** If \(X\) is an eigenvector of \(F\), then clearly \(FX \perp Y\). Suppose \(FX = \alpha X + \beta U\), where \(\beta \neq 0\) and \(U (\perp X)\) is a unit vector. Then

\[ FU = \beta^{-1}(PX + (h - \alpha)FX) = \beta X + \gamma U \]

where \(h = \alpha + \gamma\), \(\beta^2 = \alpha \gamma + \sigma\). It follows that \(PU = \sigma U\). Hence \(W \perp Y\) and so \(FX \perp Y\). \(\Box\)

**Theorem 3.4.** There does not exist any \((\eta, \eta_\alpha, \theta)\)-Einstein real hypersurface with constant coefficients functions in \(G_2(C^{m+2})\) such that \(\xi\) is tangent to \(\mathfrak{D}\).
Proof. Suppose such a real hypersurface $M$ exists. Then $\eta(\xi_a) = 0$ and $A\phi \xi_a = 0$, $a \in \{1, 2, 3\}$, by Lemma 2.2. It follows from (1.1) and (2.7) that

$$P = (4m + 7 - f_1)I - (3 + f_2)\xi \otimes \eta - (3 + f_3) \sum_{a=1}^{3} \xi \otimes \eta_a - \sum_{a=1}^{3} \phi \xi_a \otimes \eta_a \phi$$

where $P = A^2 - hA$. Since $A\phi \xi_a = 0$, we have $0 = P\phi \xi_a = 4m + 8 - f_1$. Hence $f_1 = 4m + 8$ and

$$P = -I - (3 + f_2)\xi \otimes \eta - (3 + f_3) \sum_{a=1}^{3} \xi \otimes \eta_a - \sum_{a=1}^{3} \phi \xi_a \otimes \eta_a \phi.$$  

It follows that

$$\begin{align*}
PX &= -X, \quad X \in \mathcal{H} \\
P\xi &= -(4 + f_2)\xi \\
P\xi_a &= -(4 + f_3)\xi_a.
\end{align*}$$

(3.1)

By (3.1), we see that at each point, $M$ has, at most, six distinct principal curvatures where each of them is a solution of one of the following equations:

$$\begin{align*}
z^2 - hz + 1 &= 0 \\
(3.3)
z^2 - hz + 4 + f_2 &= 0 \\
(3.4)z^2 - hz + 4 + f_3 &= 0.
\end{align*}$$

Now we consider the maximal open dense subset $M_0 \subset M$ such that the multiplicities of the principal curvatures of $M$ are constant on each connected component of $M_0$. For each principal curvature $\lambda$, denote by $\mathcal{Y}_\lambda$ the distribution on $M_0$ foliated by principal directions corresponding to $\lambda$.

We shall consider four cases: (i) $-3 \neq f_2 \neq f_3$, (ii) $-3 = f_2 \neq f_3$, (iii) $f_2 = f_3 \neq 0$, (iv) $f_2 = f_3 = 0$.

Case (i) $-3 \neq f_2 \neq f_3$. It is clear that $\xi \in \mathcal{Y}_\lambda$ for the principal curvature $\lambda$ satisfying (3.3) by virtue of Lemma 3.3. Hence $\xi$ is principal on $M_0$.

Case (ii) $-3 = f_2 \neq f_3$. In this case, $\mathcal{H} \oplus \Re \xi$ is invariant under $A$. If $\mathcal{H} \oplus \Re \xi = \mathcal{Y}_\lambda$ for a principal curvature $\lambda$ satisfying (3.2), then $M$ is Hopf. Hence, we assume that
\( \mathcal{H} \oplus \mathbb{R} \xi = \mathcal{I}_\lambda \oplus \mathcal{I}_\mu \), where \( \lambda \), \( \mu \) are two distinct solutions for (3.2). By applying Lemma 3.2, we obtain \( \xi \) is principal on \( M_0 \).

Case (iii) \( f_2 = f_3 \neq 0 \). Under this hypothesis, Lemma 3.1 gives

\[
0 = \phi \mathcal{A} \xi + \sum_{a=1}^{3} \phi_a \mathcal{A} \xi_a
= 2 \sum_{a=1}^{3} g(A \xi, \xi_a) \phi_a \xi_a + \phi(A \xi)^{2\xi} + \sum_{a,b=1}^{3} g(A \xi_a, \xi_b) \phi_a \phi_b + \sum_{a=1}^{2} \phi_a (A \xi_a)^{2\xi}
\]

where \( X^{2\xi} \) denotes the projection of \( X \) onto \( \mathcal{H} \). Note that the second and forth terms are tangent to \( \mathcal{H} \), the third term is tangent to \( \mathcal{D}^\perp \) and the first term is tangent to \( \phi \mathcal{D}^\perp \), we obtain \( g(A \xi, \xi_a) = 0 \). Consequently, \( \mathcal{H} \oplus \mathbb{R} \xi \) is invariant under \( A \). With the same argument as in the preceding case, we obtain \( \xi \) is principal on \( M_0 \).

Case (iv) \( f_2 = f_3 = 0 \).

In this case, \( \mathcal{H} \) and \( \mathcal{D}^\perp \oplus \mathbb{R} \xi \) are both invariant under \( A \). Suppose that \( \xi \) is not principal on an open subset \( G \) of \( M_0 \). Then by a suitable choice of orthonormal frame \( \{\xi_1, \xi_2, \xi_3\} \) on \( \mathcal{D}^\perp \), we may write

\[
A \xi = \alpha \xi + \beta \xi_3
\]

with \( \beta \neq 0 \). By Lemma 3.3, we obtain

\[
A \xi_3 = \beta \xi + \gamma \xi_3
\]

where \( h = \alpha + \gamma \) and \( \beta^2 = \alpha \beta - 4 \). These imply that \( \mathbb{R} \xi \oplus \mathbb{R} \xi_3 \) is invariant under \( A \). Hence, by applying suitable orthogonal transformation, we obtain

\[
A(a_j \xi + b_j \xi_3) = \alpha_j (a_j \xi + b_j \xi_3), \quad j \in \{1, 2\}
\]

\[
A \xi_1 = \alpha_1 \xi_1
\]

where \( a_j^2 + b_j^2 = 1 \), \( a_j b_j \neq 0 \) and \( a_1 a_2 + b_1 b_2 = 0 \).

Firstly, suppose that \( \mathcal{H} = \mathcal{I}_\lambda \), where \( \lambda \) is a solution for (3.2). Then we have

\[
g((\nabla X A)(a_j \xi + b_j \xi_3), Y) = g(\alpha_j (a_j \phi + b_j \phi_3) AX - A(a_j \phi + b_j \phi_3) AX, Y)
= (\alpha_j \lambda - \lambda^2) g((a_j \phi + b_j \phi_3) X, Y)
\]
for any $X, Y \in \mathcal{H}$. Hence it follows from the Codazzi equation that

$$0 = -g((\nabla_X A)Y - (\nabla_Y A)X, a_j \xi + b_j \xi_3) - 2g((a_j \phi + b_j \phi_3)X, Y)$$

$$= (\lambda^2 - \alpha_j \lambda - 1)g((a_j \phi + b_j \phi_3)X, Y).$$

Since $(a_j \phi + b_j \phi_3)\xi_3(1) = (a_j - b_j)\phi|\xi_3(1)$ and $(a_j \phi + b_j \phi_3)|\xi_3(-1) = (a_j + b_j)\phi|\xi_3(-1)$, we have

$$\lambda^2 - \alpha_j \lambda - 1 = 0, \quad j \in \{1, 2\}.$$

This implies that $\alpha_1 = \alpha_2$ and so $\xi$ is principal on $G$, which is a contradiction.

Next, we consider the case $\mathcal{H} = \mathfrak{T}_\lambda \oplus \mathfrak{T}_\mu$, where $\lambda, \mu$ are two distinct solutions of (3.2). We shall show that this case indeed cannot occur too. With a similar calculation on $g((\nabla_X A)Y - (\nabla_Y A)X, \xi_1) = -2g(\phi_1 X, Y)$, where $X, Y \in \mathcal{H}$, we obtain

$$(3.5) \quad 2A \phi_1 AX - \alpha_1 (\phi_1 A + A \phi_1) X - 2\phi_1 X = 0$$

for any $X \in \mathcal{H}$. It follows that $(A \phi_1 A \phi_1 - \phi_1 A \phi_1 A)|\mathcal{H} = 0$. Since $\mathcal{H}$ is invariant under both $A$ and $\phi_1 A \phi_1$, there exists at each point of $G$ an orthonormal basis $\{X_1, \cdots, X_{2m-4}, \phi_1 X_1, \cdots, \phi_1 X_{2m-4}\}$ on $\mathcal{H}$ in which each of them is a principal direction. If there exists $X_j$ such that $AX_j = \lambda X_j$ and $A \phi_1 X_j = \mu \phi_1 X_j$, then (3.5) gives

$$2\lambda \mu - \alpha_1 (\lambda + \mu) - 2 = 0.$$

Since $\lambda, \mu$ are distinct solutions for (3.2), $\lambda + \mu = h$ and $\lambda \mu = 1$. It follows that $\alpha_1 h = 0$. However, this contradicts the fact that $\alpha_1$ is a solution for (3.4) with $f_3 = 0$. Hence, $\phi_1 \mathfrak{T}_\lambda \subset \mathfrak{T}_\lambda$ and $\phi_1 \mathfrak{T}_\mu \subset \mathfrak{T}_\mu$. It follows from (3.5) that

$$\lambda^2 - \alpha_1 \lambda - 1 = \mu^2 - \alpha_1 \mu - 1 = 0.$$

Hence, we have $\lambda \mu = -1$. But this contradicts the fact that $\lambda \mu = 1$. Hence, this case cannot occur.

After all the above considerations, we obtain $\xi$ is principal on $M_0$. Since $M_0$ is dense, we conclude that $M$ is Hopf. By Theorem 2.4, $M$ is an open part of a real hypersurface of type $B$. It follows from [2, Prop. 2] that $\mathcal{H} = \mathfrak{T}_\lambda \oplus \mathfrak{T}_\mu$, where $\lambda = \cot r, \mu = -\tan r, r \in [0, \pi/4]$. Since $\lambda$ and $\mu$ are not solutions of (3.2), such a real hypersurface does not exist and this completes the proof.
4 Examples of \((\eta_a, \theta)\)-Einstein real hypersurfaces

In this section, we shall show that real hypersurfaces of type \(A\) in \(G_2(C^{m+2})\) are \((\eta_a, \theta)\)-Einstein (more precisely, \((\eta_a, e)\)-Einstein).

Let \(M\) be a real hypersurface of type \(A\) in \(G_2(C^{m+2})\), that is, a tube of radius \(r\) around a totally geodesic \(G_2(C^{m+1})\). Let \(J_1 \in \mathfrak{g}\) such that \(J_1 N = JN, x \in M\). Then we have

\[
\theta = \theta_1, \quad \eta_1(\xi) = 1, \quad \eta_2(\xi) = \eta_3(\xi) = 0
\]

\[
\xi_1 = \xi = -\theta_1 \xi, \quad \xi_2 = \theta_1 \xi_2 = \phi \xi_3, \quad \xi_3 = \theta_1 \xi_3 = -\phi \xi_2.
\]

Note that, under this setting, we have \(\sum_{a=1}^{3} \phi \xi_a \otimes \eta_a \phi = \sum_{a=2}^{3} \xi_a \otimes \eta_a\). Hence, the Ricci tensor \(S\) is descended to

\[
S = hA - A^2 + (4m + 7)\mathfrak{X} + \theta - 6\xi \otimes \eta - 2 \sum_{a=2}^{3} \xi_a \otimes \eta_a.
\]

(4.1)

It follows from [2, Prop. 3] that \(M\) has constant principal curvatures

\[
\alpha = \sqrt{8} \cot(\sqrt{8}r), \quad \beta = \sqrt{2} \cot(\sqrt{2}r), \quad \lambda = -\sqrt{2} \tan(\sqrt{2}r), \quad \mu = 0
\]

where \(r \in [0, \pi/\sqrt{8}]\), and \(\Sigma_\alpha = \mathbb{R} \xi, \Sigma_\beta = \mathbb{R} \xi_2 \oplus \mathbb{R} \xi_3, \Sigma_\lambda = \mathfrak{H}_1(-1), \Sigma_\mu = \mathfrak{H}_1(1)\). Note that

\[
\beta + \lambda = \alpha, \quad \beta \lambda = -2,
\]

\[
h = \alpha + 2\beta + (2m - 2)\lambda = 3\beta + (2m - 1)\lambda.
\]

We set

\[
\begin{cases}
  f_1 = 4m + 4 + (m - 1)\lambda^2 = 4m + 4 + 2(m - 1)\tan^2 \sqrt{2}r \\
  f_2 = 0 \\
  f_3 = 2\beta^2 - 4m = 4 \cot^2 \sqrt{2}r - 4m \\
  f_4 = 4 - (m - 1)\lambda^2 = 4 - 2(m - 1)\tan^2 \sqrt{2}r.
\end{cases}
\]

(4.2)

By (4.1), we obtain the followings:

\[
SX = (4m + 8)X = f_1 X + f_4 \theta X
\]

\[
SY = (h\lambda - \lambda^2 + 4m + 6) = ((2m - 2)\lambda^2 + 4m)Y = f_1 Y + f_4 \theta Y
\]

\[
S \xi_b = (h\beta - \beta^2 + 4m + 6)\xi_b = (2\beta^2 + 8)\xi_b = f_1 \xi_b + f_3 \xi_b + f_4 \theta \xi_b
\]

\[
S \xi = (h\alpha - \alpha^2 + 4m)\xi = (2\beta^2 + (2m - 2)\lambda^2)\xi = f_1 \xi + f_2 \xi + f_4 \theta \xi
\]

for any \(X \in \mathfrak{H}_1(1), Y \in \mathfrak{H}_1(-1)\) and \(b \in \{2, 3\}\). Hence we have proved the following.

**Theorem 4.1.** Let \(M\) be a tube of radius \(r\) around a totally geodesic \(G_2(C^{m+1})\) in \(G_2(C^{m+2})\). Then \(M\) is \((\eta_a, \theta)\)-Einstein with \(f_1, f_3, f_4\) where are constants given in (4.2).
In particular, by setting $f_4 = 0$ and $f_3 = 0$ respectively in (4.2), we obtain the following.

**Corollary 4.2.** Let $M$ be a tube of radius $r$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with $\cot^2(\sqrt{2}r) = (m-1)/2$. Then $M$ is $\eta_a$-Einstein with $f_1 = 4m + 8$ and $f_3 = -2(m+1)$.

**Corollary 4.3.** Let $M$ be a tube of radius $r$ around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with $\cot^2(\sqrt{2}r) = m$. Then $M$ is $\theta$-Einstein with $f_1 = 4m + 4 + 2(m - 1)/m$ and $f_4 = 4 - 2(m - 1)/m$.

**References**


