Proceedings of The Nineteenth International Workshop on
Hermitian-Grassmannian Submanifolds and Its Applications 19(2015) 1-2

(η, η_a, θ)-Einstein real hypersurfaces in complex two-plane Grassmannians

Ruenn-Huah Lee
Institute of Mathematical Sciences,
University of Malaya, 50603 Kuala Lumpur, Malaysia
e-mail : `rhlee063@hotmail.com`

Tee-How Loo
Institute of Mathematical Sciences,
University of Malaya, 50603 Kuala Lumpur, Malaysia
e-mail : `looth@um.edu.my`

(2010 Mathematics Subject Classification : 53C40, 53C15.)

Abstract. In this paper, we introduce the notion of (η, η_a, θ)-Einstein real hypersurfaces in complex two-plane Grassmannians. We show that there does not exist any (η, η_a, θ)-Einstein real hypersurface in complex two-plane Grassmannians such that ξ is tangent to $\mathfrak{D}$. Some examples of (η_a, θ)-Einstein real hypersurfaces are given.

1 Introduction

A Riemannian manifold is said to be Einstein if the Ricci tensor S is given by $S = \rho\mathbb{I}$, where ρ is a constant. The Einstein condition can be generalized in a natural manner for those spaces with certain additional geometric structures.

An almost contact metric manifold (M, ϕ, η, ξ, g) is said to be *η-Einstein* if it satisfies $S = f_1\mathbb{I} + f_2\xi \otimes \eta$, for some functions f_1, f_2 on M. Similar notion was also introduced in almost 3-contact metric geometry. Suppose now M is a manifold with an almost 3-contact metric structure $(\phi_a, \eta_a, \xi_a, g)$, $a \in \{1, 2, 3\}$. If the Ricci tensor S satisfies $S = f_1\mathbb{I} + f_2 \sum_{a=1}^{3} \xi_a \otimes \eta_a$, where f_1, f_2 are functions on M, then M is said to be η_a-Einstein.

For a Kähler or quaternionic Kähler manifold, its real hypersurfaces (i.e., submanifolds of real codimension one) naturally inherited an almost contact metric (resp. almost 3-contact metric) structure from the Kähler (resp. quaternionic Kähler) structure of the ambient manifold. The study of real hypersurfaces in a

Key words and phrases: real hypersurfaces, complex two-plane Grassmannians.

Kähler (resp. quaternionic Kähler) manifold has become a branch of almost contact metric (resp. almost 3-contact metric) geometry.

In the case of non-flat complex space form, η-Einstein real hypersurfaces were classified in [3, 5, 9]. In the classification, we obtain that f_1, f_2 must be constant and there does not exist any Einstein real hypersurface in a non-flat complex space form.

On the other hand, η_a-Einstein real hypersurfaces in non-flat quaternionic space forms were studied in [4, 8, 10], and a complete classification of such spaces could be deduced from a result in [10]. According to their results, we see that f_1, f_2 must also be constant as in the Kählerian case. Moreover, only quaternionic projective spaces $\mathbb{H}P^m$ admit an Einstein real hypersurface, which must be a tube of radius $r \in]0, \pi/2[$ over a totally geodesic $\mathbb{H}P^{m-1}$ with $\cot^2 r = 2m$.

Remark 1.1. *η-Einstein and η_a-Einstein real hypersurfaces were studied under the name of pseudo-Einstein real hypersurfaces in the above mentioned papers. However, we shall not follow that terminology in this paper to avoid the confusion.*

A complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ has some remarkable properties and structures. The most notable one being the fact that it is the unique compact irreducible Riemannian symmetric space with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ (cf. [1]). These geometric structures induce an almost contact 3-structure (ϕ_a, ξ_a, η_a), $a \in \{1, 2, 3\}$ as well as an almost contact structure (ϕ, ξ, η) on its real hypersurfaces M. These allow us to study both η-real hypersurfaces and η_a-real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

In this paper, we introduce a "generalized Einstein" condition on real hypersurface M in $G_2(\mathbb{C}^{m+2})$, apart from the impact due to both the almost contact and almost 3-contact structures on M, which also characterizes the interaction between these two structures. A real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be (η, η_a, θ)-*Einstein* if it satisfies

$$S = f_1 \mathbb{I} + f_2 \xi \otimes \eta + f_3 \sum_{a=1}^{3} \xi_a \otimes \eta_a + f_4 \theta. \tag{1.1}$$

where f_1, f_2, f_3, f_4, called the coefficient funtions, are functions on M and θ is a symmetric $(1,1)$-tensor field on M given by $\theta := \sum_{a=1}^{3} \eta_a(\xi)(\phi\phi_a - \xi \otimes \eta_a)$. For some special cases, we say that the real hypersurface M is (η, η_a)-Einstein if $f_4 = 0$; (η_a, θ)-Einstein if $f_2 = 0$; etc.

In this paper, we shall first prove that there does not exist any (η, η_a, θ)-Einstein real hypersurface M in $G_2(\mathbb{C}^{m+2})$ with constant coefficient functions and $\xi \in \mathfrak{D}$, where $\mathfrak{D}^\perp := \operatorname{span}\{\xi_1, \xi_2, \xi_3\}$ (cf. Theorem 3.4). Next we show that real hypersurfaces of type A in $G_2(\mathbb{C}^{m+2})$ are (η_a, θ)-Einstein (cf. Theorem 4.1). With this result, we also obtain example of η_a-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$.

Remark 1.2. *(η, η_a)-Einstein real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ were considered in [11, 12].*

2 Real hypersurfaces in $G_2(\mathbb{C}^{m+2})$

In this section, we summarize and list out some important formulae as well as well-known results in the theory of real hypersurfaces in complex two-plane Grassmannians (see [2, 7, 12] for details).

Denote the set of all complex 2-dimensional linear subspaces by $G_2(\mathbb{C}^{m+2})$ with Kähler structure J and quaternionic Kähler structure $\mathfrak{J}$. Let M be a connected, oriented real hypersurface isometrically immersed in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, and N a unit normal vector field on M. Denote by g the Riemannian metric on M. A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ on $G_2(\mathbb{C}^{m+2})$ induces a local almost contact metric 3-structure $(\phi_a, \xi_a, \eta_a, g)$ on M by

$$J_a X = \phi_a X + \eta_a(X)N, \qquad J_a N = -\xi_a, \qquad \eta_a(X) = g(X, \xi_a),$$

for any $X \in TM$. It follows that

$$\begin{aligned}
&\phi_a \phi_{a+1} - \xi_a \otimes \eta_{a+1} = \phi_{a+2} = -\phi_{a+1}\phi_a + \xi_{a+1} \otimes \eta_a, \\
&\phi_a \xi_{a+1} = \xi_{a+2} = -\phi_{a+1}\xi_a
\end{aligned}$$

for $a \in \{1, 2, 3\}$. The indices in the preceding equations is taken modulo three.

Let (ϕ, ξ, η, g) be the almost contact metric structure on M induced by J, i.e.,

$$JX = \phi X + \eta(X)N, \quad JN = -\xi, \quad \eta(X) = g(X, \xi).$$

The two structures (ϕ, ξ, η, g) and $(\phi_a, \xi_a, \eta_a, g)$ are related as follows

$$\phi_a \phi - \xi_a \otimes \eta = \phi \phi_a - \xi \otimes \eta_a; \quad \phi \xi_a = \phi_a \xi.$$

Next, we denote by ∇ the Levi-Civita connection and A the shape operator on M. Then

$$(2.1) \quad \left\{ \begin{aligned}
(\nabla_X \phi)Y &= \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX \\
(\nabla_X \phi_a)Y &= \eta_a(Y)AX - g(AX, Y)\xi_a \\
&\quad + q_{a+2}(X)\phi_{a+1}Y - q_{a+1}(X)\phi_{a+2}Y \\
\nabla_X \xi_a &= \phi_a AX + q_{a+2}(X)\xi_{a+1} - q_{a+1}(X)\xi_{a+2} \\
X\eta(\xi_a) &= 2\eta_a(\phi AX) + \eta_{a+1}(\xi)q_{a+2}(X) - \eta_{a+2}(\xi)q_{a+1}(X)
\end{aligned} \right.$$

for any $X, Y \in TM$, where q_a is a 1-form on M. We define a local symmetric $(1,1)$-tensor field θ_a on M by

$$\theta_a := \phi_a \phi - \xi_a \otimes \eta.$$

Then we have the following identities

$$(2.2) \quad \left\{ \begin{aligned}
\operatorname{tr} \theta_a = \eta(\xi_a), \qquad & \theta_a^2 - \phi\xi_a \otimes \eta_a \phi = \mathbb{I} \\
\theta_a \xi = -\xi_a, \quad \theta_a \xi_a &= -\xi, \quad \theta_a \phi \xi_a = \eta(\xi_a)\phi\xi_a \\
\theta_a \xi_{a+1} &= \phi \xi_{a+2} = -\theta_{a+1}\xi_a \\
-\theta_a \phi \xi_{a+1} + \eta(\xi_{a+1})\phi\xi_a &= \xi_{a+2} = \theta_{a+1}\phi\xi_a - \eta(\xi_a)\phi\xi_{a+1}.
\end{aligned} \right.$$

Further, we can easily derive from (2.1) that

$$(2.3)\quad \begin{aligned}(\nabla_X\theta_a)Y =&(\nabla_X\phi_a)\phi Y+\phi_a(\nabla_X\phi)Y-g(\nabla_X\xi,Y)\xi_a-\eta(Y)\nabla_X\xi_a\\ =&\eta_a(\phi Y)AX-g(AX,Y)\phi\xi_a+q_{a+2}(X)\theta_{a+1}Y-q_{a+1}(X)\theta_{a+2}Y\end{aligned}$$

$$(2.4)\qquad \nabla_X\phi\xi_a=\theta_aAX+\eta_a(\xi)AX+q_{a+2}(X)\phi\xi_{a+1}-q_{a+1}(X)\phi\xi_{a+2}.$$

For each $x\in M$, we define a subspace $\mathcal{H}^\perp$ of T_xM by

$$\mathcal{H}^\perp := \mathrm{span}\{\xi,\xi_1,\xi_2,\xi_3,\phi\xi_1,\phi\xi_2,\phi\xi_3\}.$$

Let $\mathcal{H}$ be the orthogonal complement of $\mathcal{H}^\perp$ in T_xM. Then $\dim\mathcal{H}=4m-4$ (resp. $\dim\mathcal{H}=4m-8$) when $\xi\in\mathfrak{D}^\perp$ (reps. $\xi\notin\mathfrak{D}^\perp$) and $\mathcal{H}$ is invariant under ϕ,ϕ_a and θ_a. Moreover, $\theta_{a|\mathcal{H}}$ has two eigenvalues: 1 and -1. Denote by $\mathcal{H}_a(\varepsilon)$ the eigenspace corresponds to the eigenvalue ε of $\theta_{a|\mathcal{H}}$. Then $\dim\mathcal{H}_a(1)=\dim\mathcal{H}_a(-1)$ is even, and

$$\begin{aligned}&\phi\mathcal{H}_a(\varepsilon)=\phi_a\mathcal{H}_a(\varepsilon)=\theta_a\mathcal{H}_a(\varepsilon)=\mathcal{H}_a(\varepsilon)\\ &\phi_b\mathcal{H}_a(\varepsilon)=\theta_b\mathcal{H}_a(\varepsilon)=\mathcal{H}_a(-\varepsilon),\quad (a\neq b).\end{aligned}$$

Now we define $\theta:=\sum_{a=1}^3\eta_a(\xi)\theta_a$, $\xi^\perp:=\sum_{a=1}^3\eta_a(\xi)\xi_a$ and $\eta^\perp:=\sum_{a=1}^3\eta_a(\xi)\eta_a$. Then by (2.2) and (2.3), we have

$$(2.5)\qquad \mathrm{tr}\,\theta=\sum_{a=1}^3\eta_a(\xi)^2=||\xi^\perp||^2$$

$$(2.6)\quad \begin{aligned}(\nabla_X\theta)Y=&\sum_{a=1}^3\{(X\eta_a(\xi))\theta_aY+\eta_a(\xi)(\nabla_X\theta_a)Y\}\\ =&\sum_{a=1}^3\{-2g(A\phi\xi_a,X)\theta_aY+\eta_a(\xi)\eta_a(\phi Y)AX-\eta_a(\xi)g(AX,Y)\phi\xi_a\}\\ =&\eta^\perp(\phi Y)AX-g(AX,Y)\phi\xi^\perp-2\sum_{a=1}^3g(A\phi\xi_a,X)\theta_aY.\end{aligned}$$

It follows from (2.5) that the tensor field θ provides an index to measure ξ for being tangential to $\mathfrak{D}$ or $\mathfrak{D}^\perp$.

Lemma 2.1. *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$. Then $0\le\mathrm{tr}\,\theta\le 1$. Moreover, we have*

(a) $\mathrm{tr}\,\theta=0$ if and only if $\xi\in\mathfrak{D}$; and

(a) $\mathrm{tr}\,\theta=1$ if and only if $\xi\in\mathfrak{D}^\perp$.

The equations of Gauss and Codazzzi are given by

$$\begin{aligned}R(X,Y)Z =& g(Y,Z)X - g(X,Z)Y + g(AY,Z)AX - g(AX,Z)AY \\ &+ g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y - 2g(\phi X,Y)\phi Z \\ &+ \sum_{a=1}^{3}\{g(\phi_a Y,Z)\phi_a X - g(\phi_a X,Z)\phi_a Y - 2g(\phi_a X,Y)\phi_a Z \\ &+ g(\theta_a Y,Z)\theta_a X - g(\theta_a X,Z)\theta_a Y\}.\end{aligned}$$

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X =& \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi \\ &+ \sum_{a=1}^{3}(\eta_a(X)\phi_a Y - \eta_a(Y)\phi_a X - 2g(\phi_a X,Y)\xi_a \\ &+ \eta_a(\phi X)\theta_a Y - \eta_a(\phi Y)\theta_a X).\end{aligned}$$

By the Gauss equation, the Ricci tensor S is given by

$$S = hA - A^2 + (4m+7)\mathbb{I} + \theta - 3\xi\otimes\eta - \sum_{a=1}^{3}(3\xi_a\otimes\eta_a + \phi\xi_a\otimes\eta_a\phi), \tag{2.7}$$

where $h := \operatorname{tr} A$ is the mean curvature of M.

Finally we state some well-known results.

Lemma 2.2 ([7]). *Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If ξ is tangent to $\mathfrak{D}$, then $A\phi\xi_a = 0$, for $a \in \{1,2,3\}$.*

Theorem 2.3 ([2]). *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both ξ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ of $G_2(\mathbb{C}^{m+2})$, or

(B) m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.

We say that a real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is of *type A* if it satisfies the first property in the characterization theorem given above. On the other hand, M is said to be of *type B* if it satisfies all properties in part (B). A connected orientable real hypersurface M in $G_2(\mathbb{C}^{m+2})$ is said to be *Hopf* if the Reeb vector field ξ is invariant under the shape operator of M. The following theorem provides the sufficient conditions of being a real hypersurface of type B.

Theorem 2.4 ([6]). *Let M be a connected orientable Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a real hypersurface of type B.*

3 (η, η_a, θ)-Einstein real hypersurfaces

We shall show that there does not exist any (η, η_a, θ)-real hypersurface in $G_2(\mathbb{C}^{m+2})$ such that ξ is tangent to $\mathfrak{D}$ everywhere in this section. We begin with deriving a basic formula for such spaces.

Lemma 3.1. *Let M be a (η, η_a, θ)-Einstein in $G_2(\mathbb{C}^{m+2})$ with constant coefficient functions, i.e.,*

$$S = f_1\mathbb{I} + f_2\xi \otimes \eta + f_3 \sum_{a=1}^{3} \xi_a \otimes \eta_a + f_4\theta.$$

where f_1, f_2, f_3, f_4 are constants. Then we have

(a) $\operatorname{grad}\operatorname{tr} S = -4f_4 A\phi\xi^{\perp}$;

(b) $f_2\phi A\xi + f_3 \sum_{a=1}^{3} \phi_a A\xi_a + f_4(A - h\mathbb{I})\phi\xi^{\perp} - f_4 \sum_{a=1}^{3} \theta_a A\phi\xi_a = 0.$

Proof. By (2.5), the scalar curvature $\operatorname{tr} S$ has the form of

$$\operatorname{tr} S = (4m-1)f_1 + f_2 + 3f_3 + f_4||\xi^{\perp}||^2.$$

It follows from (2.1) that

$$X\operatorname{tr} S = 2f_4 \sum_{a=1}^{3} \eta(\xi_a) X\eta(\xi_a) = -4f_4 \sum_{a=1}^{3} \eta(\xi_a) g(A\phi\xi_a, X) = -4f_4 g(A\phi\xi^{\perp}, X).$$

Hence we obtain Statement (a). On the other hand, by using (2.1) and (2.6), we compute

$$\begin{aligned}
(\nabla_X S)Y =& f_2(g(\nabla_X\xi, Y)\xi + \eta(Y)\nabla_X\xi) + f_3 \sum_{a=1}^{3}(g(\nabla_X\xi_a, Y)\xi_a + \eta_a(Y)\nabla_X\xi_a) \\
&+ f_4(\nabla_X\theta)Y \\
=& f_2(g(\phi AX, Y)\xi + \eta(Y)\phi AX) + f_3 \sum_{a=1}^{3}(g(\phi_a AX, Y)\xi_a + \eta_a(Y)\phi_a AX) \\
&+ f_4(\eta^{\perp}(\phi Y)AX - g(AX, Y)\phi\xi^{\perp}) - 2f_4 \sum_{a=1}^{3} g(A\phi\xi_a, X)\theta_a Y.
\end{aligned}$$

Hence, by the above equation and the Schur Lemma: $2\mathrm{div}\, S = \mathrm{grad}\,\mathrm{tr}\, S$, we obtain Statement (b). $\square$

The following lemma can be obtained with the same arguments as in the proof of [3, Prop. 5.2]. We shall state without proof.

Lemma 3.2. *Let (M, ϕ, η, ξ, g) be an almost contact metric manifold. Suppose there exist a symmetric $(1,1)$-tensor field F on M, a distribution $\mathfrak{T}$ on M with $\dim \mathfrak{T} \geq 4$, and two functions λ, μ on M with $\lambda < \mu$ such that*

(a) $\xi \in \mathfrak{T}$ everywhere;

(b) $\mathfrak{T}$ is invariant under both F and ϕ;

(c) there is an orthogonal decomposition $\mathfrak{T} = \mathfrak{T}_\lambda \oplus \mathfrak{T}_\mu$ such that $FX = \lambda X$ (resp. $FY = \mu Y$) for any $X \in \mathfrak{T}_\lambda$ (resp. $Y \in \mathfrak{T}_\mu$);

(d) $(\nabla_X F)Y - (\nabla_Y F)X = \epsilon\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi + \omega(X, Y)\}$, for any $X, Y \in \mathfrak{T}$, where ϵ is a nonvanishing function and ω is a (1,2)-tensor field on M such that $\omega(X, Y) \perp \mathfrak{T}$ for any $X, Y \in \mathfrak{T}$.

Then ξ is tangent to either $\mathfrak{T}_\lambda$ or $\mathfrak{T}_\mu$ everywhere. In other words, ξ is an eigenvector for F.

Next, we give an elementary algebraic lemma.

Lemma 3.3. *Let F be a symmetric endomorphism of a finite dimensional inner product space $\mathcal{V}$ and $X, Y \in \mathcal{V}$ with $X \perp Y$. Suppose $PX = \sigma X$ and $PY = \tau Y$, where $P = F^2 - hF$ and h, σ, τ are scalars. If $\sigma \neq \tau$, then $FX \perp Y$.*

Proof. If X is an eigenvector of F, then clearly $FX \perp Y$. Suppose $FX = \alpha X + \beta U$, where $\beta \neq 0$ and U ($\perp X$) is a unit vector. Then

$$FU = \beta^{-1}(PX + (h - \alpha)FX) = \beta X + \gamma U$$

where $h = \alpha + \gamma$, $\beta^2 = \alpha\gamma + \sigma$. It follows that $PU = \sigma U$. Hence $W \perp Y$ and so $FX \perp Y$. $\square$

Theorem 3.4. *There does not exist any (η, η_a, θ)-Einstein real hypersurface with constant coefficients functions in $G_2(\mathbb{C}^{m+2})$ such that ξ is tangent to $\mathfrak{D}$.*

Proof. Suppose such a real hypersurface M exists. Then $\eta(\xi_a) = 0$ and $A\phi\xi_a = 0$, $a \in \{1,2,3\}$, by Lemma 2.2. It follows from (1.1) and (2.7) that

$$P = (4m+7-f_1)\mathbb{I} - (3+f_2)\xi \otimes \eta - (3+f_3)\sum_{a=1}^{3} \xi \otimes \eta_a - \sum_{a=1}^{3} \phi\xi_a \otimes \eta_a \phi$$

where $P = A^2 - hA$. Since $A\phi\xi_a = 0$, we have $0 = P\phi\xi_a = 4m+8-f_1$. Hence $f_1 = 4m+8$ and

$$P = -\mathbb{I} - (3+f_2)\xi \otimes \eta - (3+f_3)\sum_{a=1}^{3} \xi \otimes \eta_a - \sum_{a=1}^{3} \phi\xi_a \otimes \eta_a \phi.$$

It follows that

$$\text{(3.1)} \qquad \begin{cases} PX &= -X, \quad X \in \mathcal{H} \\ P\xi &= -(4+f_2)\xi \\ P\xi_a &= -(4+f_3)\xi_a. \end{cases}$$

By (3.1), we see that at each point, M has, at most, six distinct principal curvatures where each of them is a solution of one of the following equations:

$$z^2 - hz + 1 = 0 \tag{3.2}$$

$$z^2 - hz + 4 + f_2 = 0 \tag{3.3}$$

$$z^2 - hz + 4 + f_3 = 0. \tag{3.4}$$

Now we consider the maximal open dense subset $M_0 \subset M$ such that the multiplicities of the principal curvatures of M are constant on each connected component of M_0. For each principal curvature λ, denote by $\mathfrak{T}_\lambda$ the distribution on M_0 foliated by principal directions correpsonding to λ.

We shall consider four cases: (i) $-3 \neq f_2 \neq f_3$, (ii) $-3 = f_2 \neq f_3$, (iii) $f_2 = f_3 \neq 0$, (iv) $f_2 = f_3 = 0$.

Case (i) $-3 \neq f_2 \neq f_3$. It is clear that $\xi \in \mathfrak{T}_\lambda$ for the principal curvature λ satisfying (3.3) by virtue of Lemma 3.3. Hence ξ is principal on M_0.

Case (ii) $-3 = f_2 \neq f_3$. In this case, $\mathcal{H} \oplus \mathbb{R}\xi$ is invariant under A. If $\mathcal{H} \oplus \mathbb{R}\xi = \mathfrak{T}_\lambda$ for a principal curvature λ satisfying (3.2), then M is Hopf. Hence, we assume that

$\mathcal{H} \oplus \mathbb{R}\xi = \mathfrak{T}_\lambda \oplus \mathfrak{T}_\mu$, where λ, μ are two distinct solutions for (3.2). By applying Lemma 3.2, we obtain ξ is principal on M_0.

Case (iii) $f_2 = f_3 \neq 0$. Under this hypothesis, Lemma 3.1 gives

$$\begin{aligned} 0 &= \phi A\xi + \sum_{a=1}^{3} \phi_a A\xi_a \\ &= 2\sum_{a=1}^{3} g(A\xi, \xi_a)\phi\xi_a + \phi(A\xi)^{\mathcal{H}} + \sum_{a,b=1}^{3} g(A\xi_a, \xi_b)\phi_a\xi_b + \sum_{a=1}^{3} \phi_a(A\xi_a)^{\mathcal{H}} \end{aligned}$$

where $X^{\mathcal{H}}$ denotes the projection of X onto $\mathcal{H}$. Note that the second and forth terms are tangent to $\mathcal{H}$, the third term is tangent to $\mathfrak{D}^\perp$ and the first term is tangent to $\phi\mathfrak{D}^\perp$, we obtain $g(A\xi, \xi_a) = 0$. Consequently, $\mathcal{H} \oplus \mathbb{R}\xi$ is invariant under A. With the same argument as in the preceding case, we obtain ξ is principal on M_0. .

Case (iv) $f_2 = f_3 = 0$.

In this case, $\mathcal{H}$ and $\mathfrak{D}^\perp \oplus \mathbb{R}\xi$ are both invariant under A. Suppose that ξ is not principal on an open subset G of M_0. . Then by a suitable choice of orthonormal frame $\{\xi_1, \xi_2, \xi_3\}$ on $\mathfrak{D}^\perp$, we may write

$$A\xi = \alpha\xi + \beta\xi_3$$

with $\beta \neq 0$. By Lemma 3.3, we obtain

$$A\xi_3 = \beta\xi + \gamma\xi_3$$

where $h = \alpha + \gamma$ and $\beta^2 = \alpha\beta - 4$. These imply that $\mathbb{R}\xi \oplus \mathbb{R}\xi_3$ is invariant under A. Hence, by applying suitable orthogonal transformation, we obtain

$$\begin{aligned} A(a_j\xi + b_j\xi_3) &= \alpha_j(a_j\xi + b_j\xi_3), \quad j \in \{1, 2\} \\ A\xi_1 &= \alpha_1\xi_1 \end{aligned}$$

where $a_j^2 + b_j^2 = 1$, $a_jb_j \neq 0$ and $a_1a_2 + b_ab_2 = 0$.

Firstly, suppose that $\mathcal{H} = \mathfrak{T}_\lambda$, where λ is a solution for (3.2). Then we have

$$\begin{aligned} g((\nabla_X A)(a_j\xi + b_j\xi_3), Y) &= g(\alpha_j(a_j\phi + b_j\phi_3)AX - A(a_j\phi + b_j\phi_3)AX, Y) \\ &= (\alpha_j\lambda - \lambda^2)g((a_j\phi + b_j\phi_3)X, Y) \end{aligned}$$

for any $X, Y \in \mathcal{H}$. Hence it follows from the Codazzi equation that

$$\begin{aligned}0 &= -g((\nabla_X A)Y - (\nabla_Y A)X, a_j\xi + b_j\xi_3) - 2g((a_j\phi + b_j\phi_3)X, Y)\\ &= (\lambda^2 - \alpha_j\lambda - 1)g((a_j\phi + b_j\phi_3)X, Y).\end{aligned}$$

Since $(a_j\phi + b_j\phi_3)_{|\mathcal{H}_3(1)} = (a_j - b_j)\phi_{|\mathcal{H}_3(1)}$ and $(a_j\phi + b_j\phi_3)_{|\mathcal{H}_3(-1)} = (a_j + b_j)\phi_{|\mathcal{H}_3(-1)}$, we have

$$\lambda^2 - \alpha_j\lambda - 1 = 0, \quad j \in \{1, 2\}.$$

This implies that $\alpha_1 = \alpha_2$ and so ξ is principal on G, which is a contradiction.

Next, we consider the case $\mathcal{H} = \mathfrak{T}_\lambda \oplus \mathfrak{T}_\mu$, where λ, μ are two distinct solutions of (3.2). We shall show that this case indeed cannot occur too. With a similar calculation on $g((\nabla_X A)Y - (\nabla_Y A)X, \xi_1) = -2g(\phi_1 X, Y)$, where $X, Y \in \mathcal{H}$, we obtain

$$2A\phi_1 AX - \alpha_1(\phi_1 A + A\phi_1)X - 2\phi_1 X = 0 \tag{3.5}$$

for any $X \in \mathcal{H}$. It follows that $(A\phi_1 A\phi_1 - \phi_1 A\phi_1 A)_{|\mathcal{H}} = 0$. Since $\mathcal{H}$ is invariant under both A and $\phi_1 A\phi_1$, there exists at each point of G an orthonormal basis $\{X_1, \cdots, X_{2m-4}, \phi_1 X_1, \cdots, \phi_1 X_{2m-4}\}$ on $\mathcal{H}$ in which each of them is a principal direction. If there exists X_j such that $AX_j = \lambda X_j$ and $A\phi_1 X_j = \mu\phi_1 X_j$, then (3.5) gives

$$2\lambda\mu - \alpha_1(\lambda + \mu) - 2 = 0.$$

Since λ, μ are distinct solutions for (3.2), $\lambda + \mu = h$ and $\lambda\mu = 1$. It follows that $\alpha_1 h = 0$. However, this contradicts the fact that α_1 is a solution for (3.4) with $f_3 = 0$. Hence, $\phi_1\mathfrak{T}_\lambda \subset \mathfrak{T}_\lambda$ and $\phi_1\mathfrak{T}_\mu \subset \mathfrak{T}_\mu$. It follows from (3.5) that $\lambda^2 - \alpha_1\lambda - 1 = \mu^2 - \alpha_1\mu - 1 = 0$. Hence, we have $\lambda\mu = -1$. But this contradicts the fact that $\lambda\mu = 1$. Hence, this case cannot occur.

After all the above considerations, we obtain ξ is principal on M_0. Since M_0 is dense, we conclude that M is Hopf. By Theorem 2.4, M is an open part of a real hypersurface of type B. It follows from [2, Prop. 2] that $\mathcal{H} = \mathfrak{T}_\lambda \oplus \mathfrak{T}_\mu$, where $\lambda = \cot r$, $\mu = -\tan r$, $r \in]0, \pi/4[$. Since λ and μ are not solutions of (3.2), such a real hypersurface does not exist and this completes the proof. $\square$

4 Examples of (η_a, θ)-Einstein real hypersurfaces

In this section, we shall show that real hypersurfaces of type A in in $G_2(\mathbb{C}^{m+2})$ are (η, η_a, θ)-Einstein (more precisely, (η_a, θ)-Einstein).

Let M be a real hypersurface of type A in $G_2(\mathbb{C}^{m+2})$, that is, a tube of radius r around a totally geodesic $G_2(\mathbb{C}^{m+1})$. Let $J_1 \in \mathfrak{J}_x$ such that $J_1 N = JN$, $x \in M$. Then we have

$$\begin{gathered}\theta = \theta_1, \quad \eta_1(\xi) = 1, \quad \eta_2(\xi) = \eta_3(\xi) = 0 \\ \xi_1 = \xi = -\theta_1 \xi, \quad \xi_2 = \theta_1 \xi_2 = \phi \xi_3, \quad \xi_3 = \theta_1 \xi_3 = -\phi \xi_2.\end{gathered}$$

Note that, under this setting, we have $\sum_{a=1}^{3} \phi \xi_a \otimes \eta_a \phi = \sum_{a=2}^{3} \xi_a \otimes \eta_a$. Hence, the Ricci tensor S is descended to

$$S = hA - A^2 + (4m+7)\mathbb{I} + \theta - 6\xi \otimes \eta - 2\sum_{a=2}^{3} \xi_a \otimes \eta_a. \tag{4.1}$$

It follows from [2, Prop. 3] that M has constant principal curvatures

$$\alpha = \sqrt{8}\cot(\sqrt{8}r),\ \beta = \sqrt{2}\cot(\sqrt{2}r),\ \lambda = -\sqrt{2}\tan(\sqrt{2}r),\ \mu = 0$$

where $r \in]0, \pi/\sqrt{8}[$, and $\mathfrak{T}_\alpha = \mathbb{R}\xi$, $\mathfrak{T}_\beta = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3$, $\mathfrak{T}_\lambda = \mathcal{H}_1(-1)$, $\mathfrak{T}_\mu = \mathcal{H}_1(1)$. Note that

$$\begin{gathered}\beta + \lambda = \alpha, \quad \beta\lambda = -2, \\ h = \alpha + 2\beta + (2m-2)\lambda = 3\beta + (2m-1)\lambda.\end{gathered}$$

We set

$$\left\{\begin{aligned} f_1 = {} & 4m + 4 + (m-1)\lambda^2 = 4m + 4 + 2(m-1)\tan^2\sqrt{2}r \\ f_2 = {} & 0 \\ f_3 = {} & 2\beta^2 - 4m = 4\cot^2\sqrt{2}r - 4m \\ f_4 = {} & 4 - (m-1)\lambda^2 = 4 - 2(m-1)\tan^2\sqrt{2}r. \end{aligned}\right. \tag{4.2}$$

By (4.1), we obtain the followings:

$$\begin{aligned} SX &= (4m+8)X = f_1 X + f_4 \theta X \\ SY &= (h\lambda - \lambda^2 + 4m + 6) = ((2m-2)\lambda^2 + 4m)Y = f_1 Y + f_4 \theta Y \\ S\xi_b &= (h\beta - \beta^2 + 4m + 6)\xi_b = (2\beta^2 + 8)\xi_b = f_1 \xi_b + f_3 \xi_b + f_4 \theta \xi_b \\ S\xi &= (h\alpha - \alpha^2 + 4m)\xi = (2\beta^2 + (2m-2)\lambda^2)\xi = f_1 \xi + f_3 \xi + f_4 \theta \xi \end{aligned}$$

for any $X \in \mathcal{H}_1(1)$, $Y \in \mathcal{H}_1(-1)$ and $b \in \{2, 3\}$. Hence we have proved the following.

Theorem 4.1. *Let M be a tube of radius r around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Then M is (η_a, θ)-Einstein with f_1, f_3, f_4 where are constants given in (4.2).*

In particular, by setting $f_4 = 0$ and $f_3 = 0$ respectively in (4.2), we obtain the following.

Corollary 4.2. *Let M be a tube of radius r around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with $\cot^2(\sqrt{2}r) = (m-1)/2$. Then M is η_a-Einstein with $f_1 = 4m+8$ and $f_3 = -2(m+1)$.*

Corollary 4.3. *Let M be a tube of radius r around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ with $\cot^2(\sqrt{2}r) = m$. Then M is θ-Einstein with $f_1 = 4m+4+2(m-1)/m$ and $f_4 = 4-2(m-1)/m$.*

References

[1] J. Berndt, *Riemannian geometry of complex two-plane Grassmannians*, Rend. Semin. Mat. Univ. Politec. Torino **55**(1997), 19–83.

[2] J. Berndt and Y.J. Suh, *Real hypersurfaces in complex two-plane Grassmannians*, Monatsh. Math. **127**(1999), 1–14.

[3] T.E. Cecil and P.J. Ryan, *Focal sets and real hypersurfaces in complex projective space*, Trans. Am. Math. Soc. **269**(1982), 481–499.

[4] T. Hamada, *A characterization of pseudo-Einstein real hypersurfaces in a quaternionic projective space*, Tsukuba J. Math. **20**(1996), 219–224.

[5] M. Kon, *Pseudo-Einstein real hypersurfaces of complex projective spaces*, J. Diff. Geom. **14**(1979), 339–354.

[6] H. Lee and Y.J. Suh, *Real hypersurfaces of type B in complex two-plane Grassmannians related to the Reeb vector*, Bull. Korean Math. Soc. **47**(2010), 551–561.

[7] T.H. Loo, *Semi-parallel real hypersurfaces in complex two-plane Grassmannians*, Differ. Geom. Appl. **24**(2014), 87–102.

[8] A. Martínez and J.D. Pérez, *Real hypersurfaces in quaternionic projective space*, Ann. Mat. Pura Appl. **145**(1986), 355–384.

[9] S. Montiel, *Real hypersurfaces of a complex hyperbolic space*, J. Math. Soc. Jpn. **37**(1985), 515–535.

[10] M. Ortega and J.D. Pérez, *D-Einstein real hypersurfaces of quaternionic space forms*, Ann. Mat. Pura Appl. **178**(2000), 33–44.

[11] J.D. Pérez, Y.J. Suh and Y. Watanabe *Generalized Einstein real hypersurfaces in complex two-plane Grassmannians*, J. Geom. Phys. **60**(2010), 1806–1818.

[12] Y.J. Suh, *Pseudo-Einstein real hypersurfaces in complex two-plane Grassmannians*, Bull. Aust. Math. Soc. **73**(2008), 183–200.