Effects of Attenuation to the Determination of Speed of Sound from Ultrasonic Interferometry

Ching Koon Yau and Christopher G. Jesudason

UNIVERSITY OF MALAYA

Department of Chemistry,
University of Malaya,
Kuala Lumpur, MALAYSIA

1. Background

Speed of sound is given by Newton–Laplace equation:

$$v \equiv \sqrt{B_a/\rho},$$

where B_a is the adiabatic bulk modulus $B_a \equiv -V\,(\partial P/\partial V)_S$ and ρ the equilibrium density. Isentropic compressibility, κ_S

$$\kappa_S \equiv \frac{1}{B_a} = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S,$$

is a thermodynamic quantity commonly determined from speed of sound measurements, notably from ultrasonic interferometry.

2. Current Theories

Standing waves

The standing wave is the result of the superposition of the forward-moving sound wave, which will be reflected on the reflector, and the backward-moving sound wave produced from the reflection. Amplitude of wave decays exponentially as a function of path length ($e^{-\alpha r}$). Each reflection at reflector reduces the amplitude by a reflection coefficient $\gamma < 1$. The particle velocity is [1]

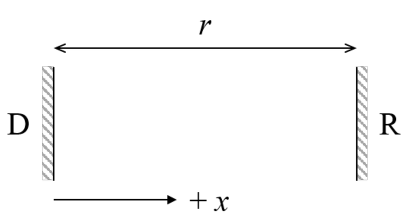

Fig. 1

$$\dot{\xi}_x = \dot{\xi}_0 e^{i\omega t} \sum_{m=0}^{\infty} \left(\gamma^m e^{-(\alpha + i\omega/v)(2mr+x)} - \gamma^{m+1} e^{-(\alpha+i\omega/v)[2(m+1)r-x]}\right), \tag{1}$$

and the excess pressure is

$$p_x = \rho v \dot{\xi} P_x + \rho v \omega \xi Q_x, \tag{2}$$

$$P_x = \frac{\left[e^{-\alpha x} - \gamma^2 e^{-\alpha(4r-x)}\right]\cos kx + \gamma\left[e^{-\alpha(2r-x)} - \gamma\, e^{-\alpha(2r+x)}\right]\cos k\,(2r-x)}{1 - 2\gamma e^{-2\alpha r}\cos 2kr + \gamma^2 e^{-4\alpha r}},$$

$$Q_x = \frac{\left[e^{-\alpha x} + \gamma^2 e^{-\alpha(4r-x)}\right]\sin kx + \gamma\left[e^{-\alpha(2r-x)} + \gamma\, e^{-\alpha(2r+x)}\right]\sin k\,(2r-x)}{1 - 2\gamma e^{-2\alpha r}\cos 2kr + \gamma^2 e^{-4\alpha r}}.$$

$i = \sqrt{-1}$, $k = \omega/v$ = angular wave number, v = speed of sound, $\dot{\xi}_0$ = particle velocity amplitude, ω = angular frequency, α = attenuation factor.

Equivalent electrical network

Hubbard [1] modelled the effect of the wave on the piezoelectric crystal against an equivalent LRC–circuit. The displacement of the crystal surface

$$M\ddot{\xi} + N\dot{\xi} + G\xi = F - Ap, \tag{3}$$

where F is the applied potential difference, A the surface area of the crystal and p is given by eqn. (2). Solving eqn. (3) in a more complex circuit gives

$$i_0^2 \propto \frac{(1+SP)^2 + (SQ)^2}{(1+SP+C)^2 + (SQ)^2}, \tag{4}$$

in which the plots [2] are given in Fig. 3 with $\sigma^2 \propto i_0^2$.

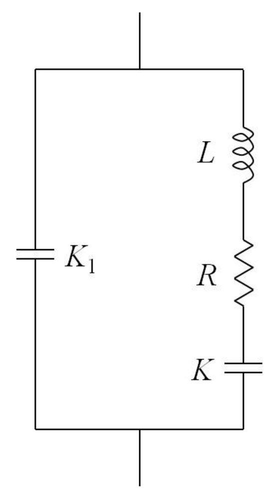

Fig. 2

$M = L/B$, $N = R/B$, $G = 1/BK$ where $B = \left(4\epsilon_{11}^2 l_z^2\right)^{-1}$. ϵ_{11} and l_z are constants related to the crystal. $S = AB\rho v/R$ and $C = [R\omega\phi_1\,(C + K_1)]^{-1}$.

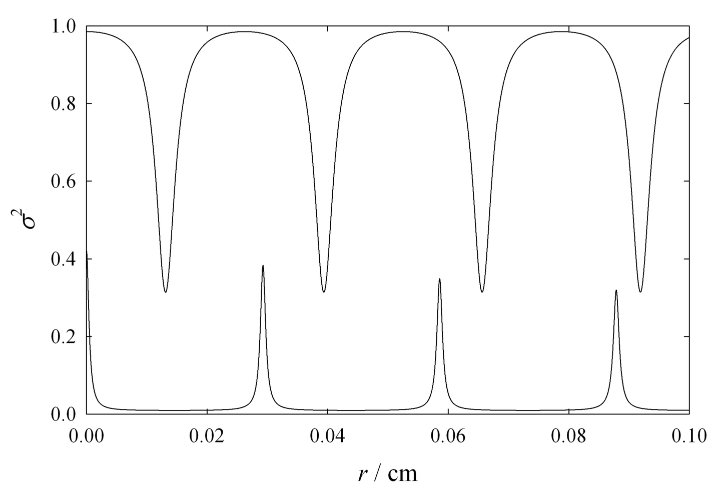

Fig. 3 The curve on top is produced for distilled water and bottom for carbon dioxide.

[1] J. C. Hubbard, *Phys. Rev.*, **38**, 1011 (1931). [2] F. E. Fox, *Phys. Rev.*, **52**, 973 (1937).

3. Our Investigation

Piezoelectric current

Hubbard did not consider a second reflection at the detector. If γ_1 is the reflection coefficient at reflector (R) (which is equivalent to Hubbard's γ) and γ_2 is the reflection coefficient at detector (D), then eqn. (1) should have been

$$\dot{\xi}_x = \dot{\xi}_0 e^{i\omega t}\sum_{m=0}^{\infty}\left[(\gamma_1\gamma_2)^m e^{-(\alpha+ik)(2mr+x)} - \gamma_1^{m+1}\gamma_2^m e^{-(\alpha+ik)[2(m+1)r-x]}\right]. \tag{5}$$

The transmitted wave to the detector is the source of the piezoelectric current. The total transmitted excess pressure is

$$p_{Tt} = -\rho_{cr} v_{cr}\,(1-\gamma_2)\,\dot{\xi}_0 e^{i\omega t}\sum_{m=0}^{\infty}\left[-\gamma_1^{m+1}\gamma_2^m e^{-(\alpha+ik)[2(m+1)r-x]}\right]$$
$$\equiv \gamma_1\,(1-\gamma_2)\,p_0 e^{i\omega t}\,[f(x,r) + i g(x,r)], \tag{6}$$

where

$$f(0,r) = \frac{e^{-2\alpha r}\cos(2kr) - \gamma_1\gamma_2 e^{-4\alpha r}}{1 + \gamma_1\gamma_2 e^{-2\alpha r}\cos(2kr) - \gamma_1^2\gamma_2^2 e^{-4\alpha r}}, \tag{7}$$

$$g(0,r) = \frac{e^{-2\alpha r}\sin(2kr)}{1 + \gamma_1\gamma_2 e^{-2\alpha r}\cos(2kr) - \gamma_1^2\gamma_2^2 e^{-4\alpha r}}. \tag{8}$$

Unlike Hubbard's approach with no averaging, the piezoelectric current is here is given by the time average excess transmitted pressure, i.e. $I \propto \left\langle [\mathrm{Re}\,(p_{Tt})]^2\right\rangle^{1/2}$ and the direct integration results

$$I \propto \frac{e^{-2\alpha r}}{\left[1 - 2\gamma_1\gamma_2\, e^{-2\alpha r}\cos(2r\omega/v) + \gamma_1^2\gamma_2^2\, e^{-4\alpha r}\right]^{1/2}}. \tag{9}$$

Stationary points

The stationary points are obtained as the roots of $dI/dr = 0$ or

$$e^{-2\alpha r}\left\{\alpha + [k\sin(2kr) - \alpha\cos(2kr)]\,G\,e^{-2\alpha r}\right\} = 0, \tag{10}$$

where $G = \gamma_1\gamma_2$. Only if $\alpha \ll k$ and r is small, the stationary points are

$$r_n = \frac{n\pi}{2k} = \frac{n\lambda}{4}, \tag{11}$$

and the separation between peaks, defined as $\Delta r_n(G,\alpha,k) \equiv r_{n+2} - r_n$ is given by

$$\Delta r_n(G,\alpha,k) = r_{n+2} - r_n \approx \frac{\lambda}{2} = \Delta r_n(1,0,k). \tag{12}$$

Effect of attenuation on the separation between peaks

Eqn. (10) apparently has not been solved analytically. Numerical methods are used to investigate the effect of α on Δr_n, with different values of G and n. For water, the changes are very small and for presentation purposes, we define the difference of the separation between peaks with that of an ideal situation where $\Delta r_0(1,0,k) = \lambda/2$ as

$$R_n \equiv \Delta r_n(G,\alpha,k) - \Delta r_0(1,0,k). \tag{13}$$

4. Results

Curve fitting

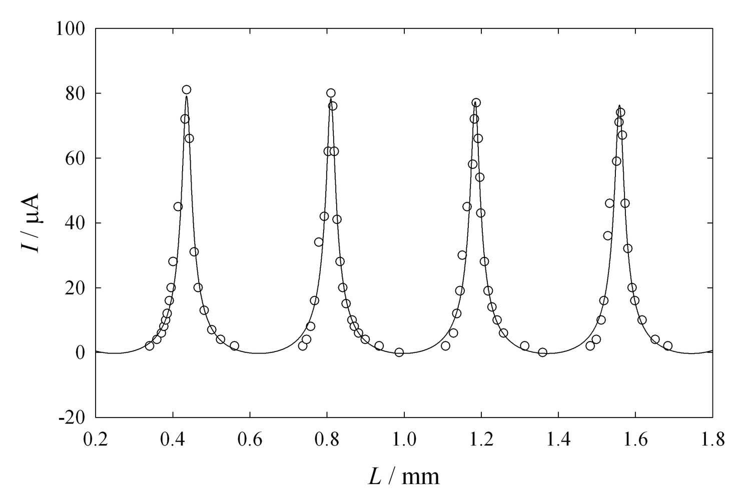

Fig. 4 The experimental points are fitted with the curve eqn. (9) with $r = L + L_0 - X$ and $I_{obs} = M\,(I + Y)$.

Variable	Value
α / mm^{-1}	2.492×10^{-3}
k / mm^{-1}	8.3993
G	0.8511
X / mm	−0.080
Y	−0.547
M / μA	16.12

Effect of attenuation on the separation between peaks

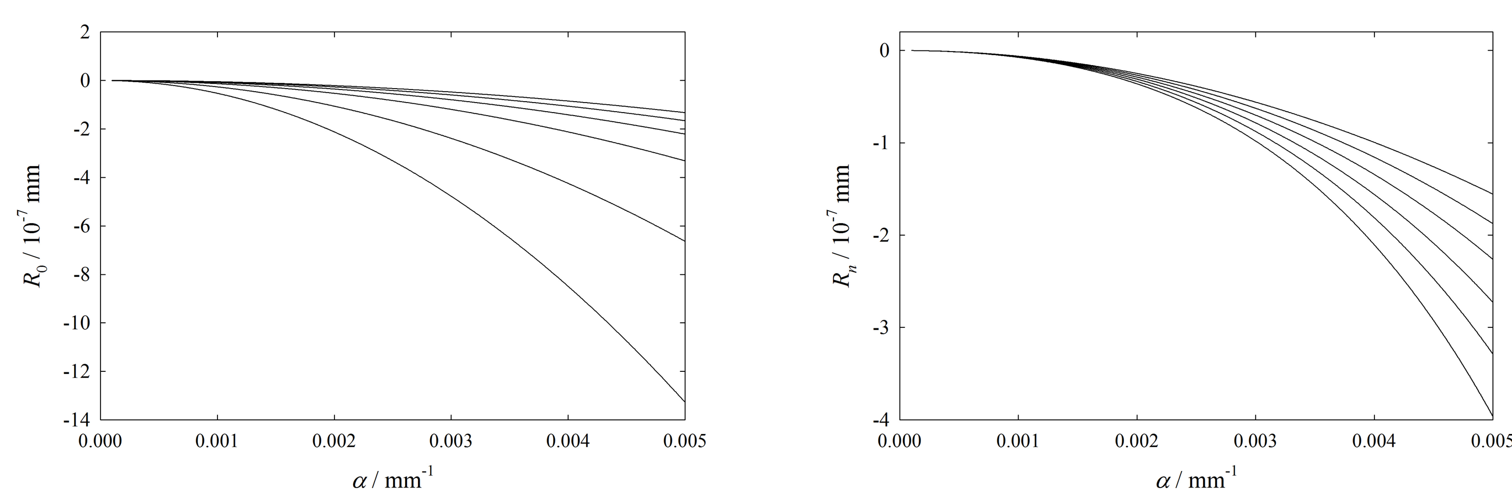

Fig. 5 *Left*: Six lines showing the variation of R_0 with α for, from bottom to top, G = 0.1, 0.2, 0.4, 0.6, 0.8 and 1.0 respectively for distilled water, in all cases with k = 8.3993 mm^{-1}. *Right*: Six lines showing the variation of R_n with α for, from top to bottom, n = 0, 100, 200, 300, 400, and 500 for distilled water, in all cases with G = 0.8511 and k = 8.3993 mm^{-1}.

5. Discussions

1. Hubbard's derivation did not include the reflection factor at the detector, γ_2. Fox subsequent measurements revealed that γ is lower than the theoretical values which may be attributed to the absence of γ_2.
2. Fox, using Hubbard's equivalent electrical network analysis, produces a non-equivalent inverted graph (compare Fig. 3 with the experimental values of Fig. 4).
3. By curve fitting, the value of k obtained gives the speed of sound in water as v = (1496.42 ± 0.09) m s^{-1} (lit. v = 1496.687 m s^{-1} [3]) while the attenuation factor is of the same order in magnitude as found in lit. [4].
4. The effect of attenuation to the separation between peaks is shown in Fig. 5. For distilled water, this effect is very small, which is typically of the order 10^{-7} mm, and therefore the separation can be practically assumed constant.

[3] V. A. Del Grosso & C. W. Mader, *J. Acoust. Soc. Am.*, **52** 1442 (1972). [4] C. E. Teeter, Jr., *J. Acoust. Soc. Am.*, **18** 488 (1946).

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CKY: yauchingkoon@gmail.com, CGJ: jesu@um.edu.my

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