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19TH CONFERENCE OF THE INTERNATIONAL LINEAR ALGEBRA SOCIETY

SEOUL, KOREA
Sungkyunkwan University
August 6-9, 2014

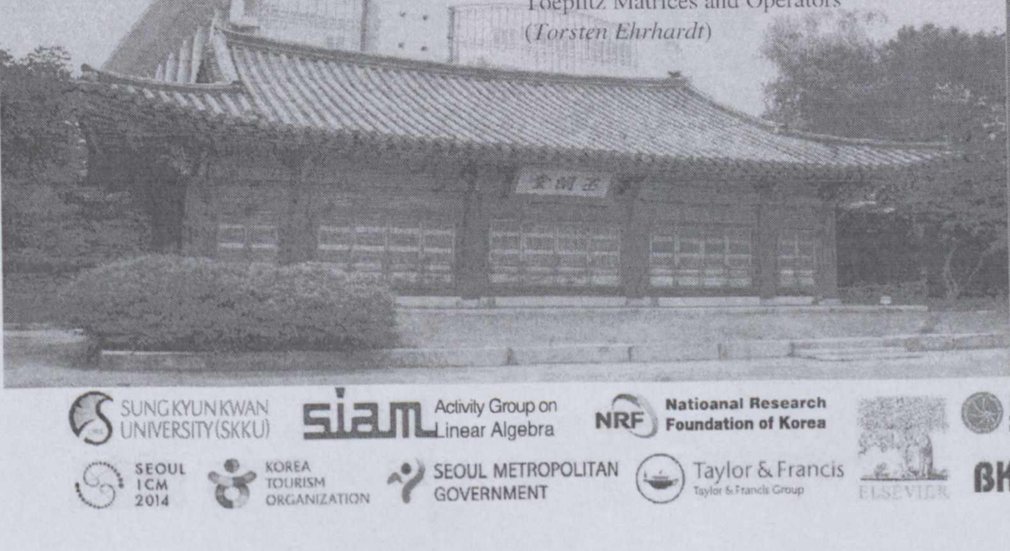
Solidarity in Linear Algebra

PLENARY SPEAKERS

Ravindra Bapat (LAMA Lecturer)
Peter Benner
Dario Bini (LAA Lecturer)
Shaun Fallat (Taussky-Todd Lecturer)
Andreas Frommer (SIAG/LA Lecturer)
Stephane Gaubert
Chi-Kwong Li
Yongdo Lim
Panayiotis Psarrakos
Vladimir Sergeichuk
Bernd Sturmfels
Tin-Yau Tam

INVITED MINISYMPOSIA

Combinatorial Problems in Linear Algebra
(Richard. A. Brualdi and Geir Dahl)
Matrix Inequalities
(Fuzhen Zhang and Minghua Lin)
Spectral Theory of Graphs and Hypergraphs
(Vladimir S. Nikiforov)
Tensor Eigenvalues
(Jia-Yu Shao and Liqun Qi)
Quantum Information and Computing
(Chi-Kwong Li and Yiu Tung Poon)
Riordan arrays and Related Topics
(Gi-Sang Cheon and Louis W. Shapiro)
Nonnegative Matrices and Generalizations
(Judi McDonald)
Toeplitz Matrices and Operators
(Torsten Ehrhardt)



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4.CONTRIBUTED TALKS(CT)

GEOMETRY OF PER-ALTERNATE TRIANGULAR MATRICES

Kiam Heong Kwa
University of Malaya, MALAYSIA
Aug 6 (Wed), 10:30–10:55, (2B, 9B208)

In this talk, we study bijective adjacency invariant maps on per-alternate upper triangular matrices over an arbitrary field. Contrary to those on full matrices, it is found that such maps not only carry rank-2 matrices to rank-2 matrices, but may also fix all rank-2 matrices.

(This is a joint work with Wai Leong Chooi and Ming Huat Lim from University of Malaya.)
Keywords : Per-alternate triangular matrices, bijective adjacency invariant maps, rank-2 preservers
E-mail : khkwa@um.edu.my

CONVEXITY OF LINEAR IMAGES OF REAL MATRICES WITH PRESCRIBED SINGULAR VALUES AND SIGN OF DETERMINANT

Pan-Shun Lau
The University of Hong Kong, HONG KONG
Aug 6 (Wed), 10:55–11:20, (2B, 9B208)

For any $s = (s_1, ..., s_n) \in \mathbb{R}^n$, let $O(s)$ denote the set

$$\{U\mathrm{diag}(s_1, ..., s_n)V : U, V \in \mathrm{SO}(n)\},$$

where $\mathrm{diag}(s_1, ..., s_n)$ is the diagonal matrix with $s_1, ..., s_n$ as diagonal entries, and $\mathrm{SO}(n)$ the set of all real orthogonal matrices of order n with positive determinant. It is clear that $O(s)$ is the set of all real $n \times n$ matrices with singular values $|s_1|, ..., |s_n|$ and their sign of determinant equal to the sign of $\prod_{i=1}^n s_i$. In this paper we consider linear maps L from $\mathbb{R}^{n\times n}$ to $\mathbb{R}^2$, and prove that for any $s \in \mathbb{R}^n$ with $n \geq 3$, the linear image $L(O(s))$ is always convex. We also give an example to show that $L(O(s))$ may fail to be convex if L is a linear map to $\mathbb{R}^3$. Our study is motivated by a result of RC Thompson which gave some necessary and sufficient conditions on the existence of a real square matrix with prescribed sign of determinant, prescribed diagonal elements and prescribed singular values. To prove our convexity result, we first consider two types of semi-group actions on $\mathbb{R}^n$ to obtain a new necessary and sufficient condition on Thompson's result. Then for $s, s' \in \mathbb{R}^n$, we apply this new condition to study inclusion relations of the form $L(O(s)) \subset L(O(s'))$ which hold for all linear maps L under consideration. Such inclusion relations are then applied to give our convexity result on $L(O(s))$. The techniques we used are motivated by a result of YT Poon which gave an elegant proof on the convexity of the c-numerical range. We also extend the results to real non-square matrices. This is a joint work with NK Tsing.
Keywords : Singular values, linear images.
E-mail : panlau@hku.hk

APPROXIMATION PROBLEMS IN THE RIEMANNIAN METRIC ON POSITIVE DEFINITE MATRICES

Rajendra Bhatia, Tanvi Jain*
Indian Statistical Institute, INDIA
Aug 6 (Wed), 11:20–11:45, (2B, 9B208)

There has been considerable work on matrix approximation problems in the space of matrices with Euclidean and unitarily invariant norms. The purpose of this talk is to initiate the study of approximation problems in the space of positive definite matrices with the Riemannian metric. In particular, we focus on the reduction of these problems to approximation problems in the space of Hermitian matrices and in Euclidean spaces.
Keywords : Matrix approximation problem, positive definite matrix, Riemannian metric, convex set, Finsler metric
E-mail : tanvi@isid.ac.in

DIRECT ALGEBRAIC SOLUTIONS TO TROPICAL OPTIMIZATION PROBLEMS

Nikolai Krivulin
Saint Petersburg State University, RUSSIA
Aug 6 (Wed), 10:30–10:55, (2B, 9B215)

Multidimensional optimization problems are considered within the framework of tropical (idempotent) algebra. The problems consist of minimizing or maximizing functions defined on vectors of a finite-dimensional semimodule over an idempotent semifield, and may have constraints in the form of linear equations and inequalities. The objective function can be either a linear function or a nonlinear function that is given by the vector operator of multiplicative conjugate transposition.
We start with an overview of known optimization problems and related solution methods. Certain problems that were originally stated in different terms, but can readily be reformulated in the tropical algebra setting, are also included.
First, we present problems that have linear objective functions and thus are idempotent analogues of those in conventional linear programming. Then, problems with nonlinear objective functions are examined, including Chebyshev and Chebyshev-like approximation problems, problems with minimization and maximization of span seminorm, and problems that involve the evaluation of the spectral radius of a matrix. Some of these problems admit complete direct solutions given in an explicit vector form. The known solutions to other problems are obtained in an indirect form of iterative algorithms that produce a particular solution if any or show that there is no solution.

Geometry of Per-alternate Triangular Matrices

Kiam Heong Kwa

Institute of Mathematical Sciences, University of Malaya, Malaysia

August 6, 2014

Abstract

In this talk, we study bijective adjacency invariant maps on per-alternate upper triangular matrices over an arbitrary field. Contrary to those on full matrices, it is found that such maps not only carry rank-2 matrices to rank-2 matrices, but may also fix all rank-2 matrices.

(This is a joint work with Wai Leong Chooi and Ming Huat Lim from University of Malaya.)

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Notation

Throughout this talk, we use the following notation.

$\mathbb{F}$: an arbitrary field.

$\mathcal{M}_{m,n}(\mathbb{F})$: the totality of $m \times n$ matrices over $\mathbb{F}$.

$\mathcal{M}_n(\mathbb{F})$: $\mathcal{M}_{n,n}(\mathbb{F})$.

$\mathcal{T}_n(\mathbb{F})$: the totality of upper triangular elements of $\mathcal{M}_n(\mathbb{F})$.

$\mathcal{PA}_n(\mathbb{F})$: the totality of per-alternate elements of $\mathcal{M}_n(\mathbb{F})$.

$\mathcal{PAT}_n(\mathbb{F})$: $\mathcal{T}_n(\mathbb{F}) \cap \mathcal{PA}_n(\mathbb{F})$.

Per-alternate triangular matrices

For any $A \in \mathcal{M}_n(\mathbb{F})$, let

$$A^+ = J_n A^T J_n,$$

where A^T is the transpose of A and J_n is the element of $\mathcal{M}_n(\mathbb{F})$ with ones on the minor diagonal and zeros elsewhere[1]:

$$J_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

An $A \in \mathcal{M}_n(\mathbb{F})$ is called **per-alternate** provided

$$A^+ = -A$$

and the minor diagonal of A vanishes. As indicated above, $\mathcal{PAT}_n(\mathbb{F})$ denotes the totality of per-alternate upper triangular matrices of order n.

[1] Indeed, A^+ can be obtained from A by reversing the rows of A followed by the columns and finally transposing.

Adjacency preserving

Two (distinct) elements A and B of $\mathcal{M}_n(\mathbb{F})$ are called **adjacent** provided

$$\operatorname{rank}(A - B) = 2.$$

We are interested in bijective maps $\Psi : \mathcal{PAT}_n(\mathbb{F}) \to \mathcal{PAT}_n(\mathbb{F})$ preserving adjacency in the sense that

$$\boxed{\operatorname{rank}(A - B) = 2 \text{ if and only if } \operatorname{rank}(\Psi(A) - \Psi(B)) = 2}$$

for any two (distinct) $A, B \in \mathcal{PAT}_n(\mathbb{F})$. These maps are called **adjacency preserving bijections**. For simplicity, it is assumed that these maps carry 0 to 0.

Standard adjacency preserving bijections

Standard adjacency preserving bijections abound. For instance, if $n \geq 4$, then for any $\alpha \in \mathbb{F}\backslash 0$ and any invertible $P \in \mathcal{T}_n(\mathbb{F})$,

$$\Psi(A) = \alpha P \sigma(A) P^+ \quad \forall A \in \mathcal{PAT}_n(\mathbb{F}),$$

where σ is either the identity $A \mapsto A$ or the map $A \to A^+ = J_n A^T J_n$, is an adjacency preserving bijection.

Adjacency preserving maps induced by strictly triangular per-alternate matrices

Each per-alternate strictly triangular matrix induces uniquely an adjacency preserving bijection distinct from the standard ones. Explicitly,

Theorem

Let $X \in \mathcal{PAT}_n(\mathbb{F})$ be strictly triangular, where $n \geq 4$. Then the map

$$\boxed{\Psi_X(A) = \sum_{k=0}^{n-2} A(XA)^k = A + AXA + \cdots + A(XA)^{n-2}}$$

is an adjacency preserving bijection on $\mathcal{PAT}_n(\mathbb{F})$. Furthermore, if $Y \in \mathcal{PAT}_n(\mathbb{F})$ is strictly triangular, then $\Psi_X = \Psi_Y$ only if $X = Y$.

As far as the authors can ascertain, it is not obvious that such a map Ψ_X is an adjacency preserving bijection. It is the goal of this talk to elucidate this. We do this by studying the "constituents" of Ψ_X.

A commutative group of adjacency preserving maps

The collection of maps

$$\{\Psi_X | X \text{ is a strictly triangular element of } \mathcal{PAT}_n(\mathbb{F})\}$$

is a commutative group under composition of maps isomorphic to the group of strictly triangular elements of $\mathcal{PAT}_n(\mathbb{F})$ under matrix addition through the isomorphism

$$\boxed{X \mapsto \Psi_X.}$$

Generating matrices and generators

We need the collection of *triangular pairs* of indices of order n, i.e., the set

$$\Delta_n = \{(i,j) \in \mathbb{N} \times \mathbb{N} | i < j < n - i + 1\}.$$

For any $\alpha \in \mathbb{F}$ and each $(i,j) \in \Delta_n$, let

$$\boxed{X_{\alpha,i,j} = \alpha(E_{i,j} - E_{i,j}^{+}),}$$

where $E_{i,j}$ is the element of $\mathcal{M}_n(\mathbb{F})$ with one on the (i,j)th entry and zeros elsewhere, and let

$$\boxed{\Psi_{\alpha,i,j}(A) = A + AX_{\alpha,i,j}A \quad \forall A \in \mathcal{PAT}_n(\mathbb{F}).}$$

Note that $\Psi_{\alpha,i,j} = \Psi_{X_{\alpha,i,j}}$. For ease of reference, such a matrix $X_{\alpha,i,j}$ is called a **generating matrix** and the map $\Psi_{\alpha,i,j}$ a **generator**.

Decomposition into generators

Lemma (Decomposition Lemma)

For each strictly triangular $X \in \mathcal{PAT}_n(\mathbb{F})$, let

$$X = \sum_{(i,j)\in\Delta_n} X_{\alpha_{i,j},i,j}$$

be the unique decomposition of X into generating matrices $X_{\alpha_{i,j},i,j}$ for some $\{\alpha_{i,j}\}_{(i,j)\in\Delta_n} \subset \mathbb{F}$. Then

$$\boxed{\Psi_X = \sum_{k=0}^{n-2} A(XA)^k = \circ_{(i,j)\in\Delta_n}\Psi_{\alpha_{i,j},i,j},}$$

where $\circ$ denotes the usual composition of maps.

An example: $n = 4$

Let

$$X = \begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} & 0 \\ 0 & 0 & 0 & -\alpha_{13} \\ 0 & 0 & 0 & -\alpha_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{22} & 0 & -a_{13} \\ 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & 0 & -a_{11} \end{pmatrix}.$$

Then

$$\Psi_X(A) = A + AXA + A(XA)^2$$
$$= \begin{pmatrix} a_{11} & a_{11}a_{22}\alpha_{12} + a_{12} & -a_{11}a_{22}\alpha_{13} + a_{13} & 0 \\ 0 & a_{22} & 0 & a_{11}a_{22}\alpha_{13} - a_{13} \\ 0 & 0 & -a_{22} & -a_{11}a_{22}\alpha_{12} - a_{12} \\ 0 & 0 & 0 & -a_{11} \end{pmatrix}$$

An example: $n = 4$

Decompose

$$\underbrace{\begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} & 0 \\ 0 & 0 & 0 & -\alpha_{13} \\ 0 & 0 & 0 & -\alpha_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 0 & \alpha_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\alpha_{12} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{X_{\alpha_{1,2},1,2}} + \underbrace{\begin{pmatrix} 0 & 0 & \alpha_{13} & 0 \\ 0 & 0 & 0 & -\alpha_{13} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{X_{\alpha_{1,3},1,3}}.$$

An example: $n = 4$

Decomposition Lemma says

$$\Psi_X(A) = \Psi_{\alpha_{1,2},1,2} \circ \Psi_{\alpha_{1,3},1,3}(A),$$

where

$$\Psi_{\alpha_{1,2},1,2}(A) = \begin{pmatrix} a_{11} & a_{11}a_{22}\alpha_{12} + a_{12} & a_{13} & 0 \\ 0 & a_{22} & 0 & -a_{13} \\ 0 & 0 & -a_{22} & -a_{11}a_{22}\alpha_{12} - a_{12} \\ 0 & 0 & 0 & -a_{11} \end{pmatrix};$$

$$\Psi_{\alpha_{1,3},1,3}(A) = \begin{pmatrix} a_{11} & a_{12} & -a_{11}a_{22}\alpha_{13} + a_{13} & 0 \\ 0 & a_{22} & 0 & a_{11}a_{22}\alpha_{13} - a_{13} \\ 0 & 0 & -a_{22} & -a_{12} \\ 0 & 0 & 0 & -a_{11} \end{pmatrix}.$$

Do Ψ_X and Ψ_Y commute?

Is it true that for any $A \in \mathcal{PAT}_n(\mathbb{F})$,

$$\sum_{k=0}^{n-2} \Psi_X(A)(Y\Psi_X(A))^k = \Psi_Y \circ \Psi_X(A) = \Psi_X \circ \Psi_Y(A) = \sum_{k=0}^{n-2} \Psi_Y(A)(X\Psi_Y(A))^k?$$

Yes by Decomposition Lemma, if so are the generators $\Psi_{\alpha_{i,j},i,j}$'s.

Generators are pairwise commutative. That is, for any $\alpha, \beta \in \mathbb{F}$ and all $(i,j), (k,l) \in \Delta_n$,

$$\boxed{\Psi_{\alpha,i,j} \circ \Psi_{\beta,k,l} = \Psi_{\beta,k,l} \circ \Psi_{\alpha,i,j}.}$$

In addition,

$$\boxed{\Psi_{\alpha,i,j} \circ \Psi_{\beta,i,j} = \Psi_{\alpha+\beta,i,j},}$$

so that

$$\boxed{\Psi_{\alpha,i,j}^{-1} = \Psi_{-\alpha,i,j}.}$$

Ψ_X and Ψ_Y commute

If

$$X = \sum_{(i,j)\in\Delta_n} X_{\alpha_{i,j},i,j} \text{ and } Y = \sum_{(i,j)\in\Delta_n} X_{\beta_{i,j},i,j},$$

so that

$$\Psi_X = \circ_{(i,j)\in\Delta_n} \Psi_{\alpha_{i,j},i,j} \text{ and } \Psi_Y = \circ_{(i,j)\in\Delta_n} \Psi_{\beta_{i,j},i,j},$$

then

$$\boxed{\Psi_X \circ \Psi_Y = \Psi_Y \circ \Psi_X = \circ_{(i,j)\in\Delta_n} \Psi_{\alpha_{i,j}+\beta_{i,j},i,j} = \Psi_{X+Y}}$$

because

$$X + Y = \sum_{(i,j)\in\Delta_n} X_{\alpha_{i,j}+\beta_{i,j},i,j}.$$

Is Ψ_X bijective?

For any $B \in \mathcal{PAT}_n(\mathbb{F})$, is there a unique $A \in \mathcal{PAT}_n(\mathbb{F})$ such that

$$B = \Psi_X(A) = \sum_{k=0}^{n-2} A(XA)^k?$$

Not easy to address is directly. However, it follows from the preceding slide that

$$\Psi_X \circ \Psi_{-X} = \Psi_0 = \text{identity}.$$

Thus

$$\boxed{\Psi_{-X} = \Psi_X^{-1}.}$$

Is Ψ_X adjacency preserving?

Is it true that

$$\operatorname{rank}(A-B) = 2 \text{ if and only if } \operatorname{rank}(\Psi_X(A) - \Psi_X(B)) = 2?$$

Yes by Decomposition Lemma, if so are the generators $\Psi_{\alpha_{i,j},i,j}$'s.

We state some facts without proofs.

(a) For any generating matrix $X_{\alpha,i,j}$ and any $A \in \mathcal{PAT}_n(\mathbb{F})$,

$$X_{\alpha,i,j}AX_{\alpha,i,j} = 0.$$

(b) For any rank-2 $A \in \mathcal{PAT}_n(\mathbb{F})$, there is a nonsingular $P \in \mathcal{T}_n(\mathbb{F})$ and a generating matrix $X_{\beta,k,l}$ such that

$$A = PX_{\beta,k,l}P^+.$$

(c) As a consequence, for any rank-2 $Z \in \mathcal{PAT}_n(\mathbb{F})$ and any generating matrix $X_{\alpha,i,j}$,

$$ZX_{\alpha,i,j}Z = 0.$$

Ψ_X is adjacency preserving

Let $A, B \in \mathcal{PAT}_n(\mathbb{F})$ be adjacent, so that $Z = A - B$ has rank 2. Using the above-mentioned facts, it can be shown that

$$\begin{aligned}\Psi_{\alpha,i,j}(A) - \Psi_{\alpha,i,j}(B) &= A + AX_{\alpha,i,j}A - B - BX_{\alpha,i,j}B \\ &= A - B + AX_{\alpha,i,j}A - (A - Z)X_{\alpha,i,j}(A - Z) \\ &= A - B + AX_{\alpha,i,j}(A - B) + (A - B)X_{\alpha,i,j}A \\ &= (I_n + AX_{\alpha,i,j})(A - B)(I_n + X_{\alpha,i,j}A),\end{aligned}$$

where I_n is the identity matrix. Since $AX_{\alpha,i,j}$ and $X_{\alpha,i,j}A$ are strictly triangular, $\Psi_{\alpha,i,j}(A) - \Psi_{\alpha,i,j}(B)$ has the rank of $A - B$.

Standard adjacency preserving bijections
Adjacency preserving by strictly triangular matrices
Decomposition Lemma
Properties of Ψ_X's

Is Ψ_X uniquely determined by X?

Say

$$X = \sum_{(i,j)\in Delta_n} X_{\alpha_{i,j},i,j} \text{ and } Y = \sum_{(i,j)\in\Delta_n} X_{\beta_{i,j},i,j}.$$

are such that

$$\circ_{(i,j)\in\Delta_n}\Psi_{\beta_{i,j},i,j} = \Psi_Y = \Psi_X = \circ_{(i,j)\in\Delta_n}\Psi_{\alpha_{i,j},i,j}.$$

Then

$$\Psi_{Y-X} = \circ_{(i,j)\in\Delta_n}\Psi_{\beta_{i,j}-\alpha_{i,j},i,j} = \circ_{(i,j)\in\Delta_n}\Psi^{-1}_{\alpha_{i,j},i,j}\circ\Psi_{\beta_{i,j},i,j} = \Psi_X^{-1}\circ\Psi_Y = \mathrm{Id}$$

where Id stands for the identity map, because

$$Y - X = \sum_{(i,j)\in\Delta_n} \left(X_{\beta_{i,j},i,j} - X_{\alpha_{i,j},i,j}\right).$$

Ψ_X is uniquely determined by X

Since $\Psi_{Y-X} = \mathrm{Id}$, we have
$A + A(Y-X)A + \cdots + A((Y-X)A)^{n-2} = \Psi_{Y-X}(A) = A$ or

$$A(Y-X)A + A((Y-X)A)^2 + \cdots + A((Y-X)A)^{n-2} = 0$$

for all $A \in \mathcal{PAT}_n(\mathbb{F})$. Left multiplying the last equation with $A(Y-X)$, together with the fact that $A((Y-X)A)^{n-1}$ is null, yields

$$A((Y-X)A)^2 + \cdots + A((Y-X)A)^{n-2} = 0$$

for any $A \in \mathcal{PAT}_n(\mathbb{F})$. Hence the conclusion $Y = X$ follows from the fact that

$$A(Y-X)A = 0$$

for all $A \in \mathcal{PAT}_n(\mathbb{F})$ and

Lemma

A strictly triangular $Z \in \mathcal{PAT}_n(\mathbb{F})$ is such that $AZA = 0$ for all $A \in \mathcal{PAT}_n(\mathbb{F})$ if and only if $Z = 0$.

Does Ψ_X fix all rank-2 elements of $\mathcal{PAT}_n(\mathbb{F})$?

Yes by Decomposition Lemma, if so are the generators $\Psi_{\alpha_{i,j},i,j}$'s.

Say $A \in \mathcal{PAT}_n(\mathbb{F})$ is a fixed point of Ψ_X, i.e,

$$\Psi_X(A) = A + AXA + \cdots + A(XA)^{n-2} = A$$

or

$$AXA + A(XA)^2 + A(XA)^{n-2} = 0.$$

Left multiplying the above equation with AX, together with the fact that $A(XA)^{n-1}$ vanishes, yields

$$A(XA)^2 + \cdots + A(XA)^{n-2} = 0.$$

This implies that

Lemma

For any $A \in \mathcal{PAT}_n(\mathbb{F})$, $\Psi_X(A) = A$ if and only if $AXA = 0$.

Ψ_X fixes all rank-2 elements of $\mathcal{PAT}_n(\mathbb{F})$

For a generating matrix $X_{\alpha,i,j}$, recall that $ZX_{\alpha,i,j}Z = 0$ for any rank-2 $Z \in \mathcal{PAT}_n(\mathbb{F})$. Hence the generators $\Psi_{\alpha_{i,j},i,j}$'s fix rank-2 elements of $\mathcal{PAT}_n(\mathbb{F})$ and so does Ψ_X.

Proof Sketch of Decomposition Lemma

Let

$$X = \sum_{(i,j)\in\Delta_n} X_{\alpha_{i,j},i,j}$$

be the decomposition of X into generating matrices. The proof proceeds by induction on the number of nonzero generating matrices $X_{\alpha_{i,j},i,j} \neq 0$.

Basis of induction. Say $X = X_{\alpha_{i,j},i,j} \neq 0$ for some $(i,j) \in \Delta_n$. Then

$$XAX = X_{\alpha_{i,j},i,j} A X_{\alpha_{i,j},i,j} = 0$$

and whence

$$\begin{aligned}\Psi_{\alpha_{i,j},i,j}(A) &= A + AX_{\alpha_{i,j},i,j}A \\ &= \underbrace{A + AXA}_{A+AX_{\alpha_{i,j},i,j}A} + \underbrace{AXAXA + \cdots}_{0} \\ &= \sum_{k=0}^{n-2} A(XA)^k = \Psi_X(A)\ \forall A \in \mathcal{PAT}_n(\mathbb{F}).\end{aligned}$$

Proof Sketch of Decomposition Lemma

Induction hypothesis. Suppose the assertion holds for all such maps Ψ_Y induced by strictly triangular $Y = \sum_{(i,j)\in\Delta_n} X_{\beta_{i,j},i,j} \in \mathcal{PAT}_n(\mathbb{F})$ having $p-1$ nonzero generating matrices $X_{\beta_{i,j},i,j}$'s for some $p \geq 2$.

Induction step. Say X has p nonzero generating matrices, so that

$$X = \sum_{k=1}^{p} X_{\alpha_k,i_k,j_k}$$

for some $\{(i_k,j_k)\}_{k=1}^{p} \subset \Delta_n$ and $\{\alpha_k\}_{k=1}^{p} \subset \mathbb{F}$. Let

$$Y = \sum_{k=1}^{p-1} X_{\alpha_k,i_k,j_k}.$$

Then, by induction hypothesis, for all $A \in \mathcal{PAT}_n(\mathbb{F})$,

$$\circ_{k=1}^{p-1} \Psi_{\alpha_k,i_k,j_k}(A) = \sum_{k=0}^{n-2} A(YA)^k = \Psi_Y(A).$$

Standard adjacency preserving bijections
Adjacency preserving by strictly triangular matrices
Decomposition Lemma
Properties of Ψ_X's

Proof Sketch of Decomposition Lemma

Thus

$$\Psi_{\alpha_p,i_p,j_p} \circ \Psi_Y(A) = \sum_{k=0}^{n-2} A(YA)^k + \sum_{k=0}^{n-2} A(YA)^k X_{\alpha_p,i_p,j_p} \sum_{k=0}^{n-2} A(YA)^k$$

for all $A \in \mathcal{PAT}_n(\mathbb{F})$. Since $A(YA)^{n-1} = 0$,

$$\sum_{i=0}^{n-2} A(YA)^k X_{\alpha_p,i_p,j_p} \sum_{k=0}^{n-2} A(YA)^k = \sum_{k=1}^{n-1}\sum_{l=0}^{k-1} A(YA)^l X_{\alpha_p,i_p,j_p} A(YA)^{k-l-1}.$$

Proof Sketch of Decomposition Lemma

Explicitly,

$$
\begin{aligned}
\sum_{i=0}^{n-2} A(YA)^k X_{\alpha_p,i_p,j_p} \sum_{k=0}^{n-2} A(YA)^k &= \sum_{k,l=0}^{n-2} A(YA)^k X_{\alpha_p,i_p,j_p} A(YA)^l \\
&= \sum_{\substack{k+l=0 \\ k\geq 0,\, l\geq 0}}^{n-2} A(YA)^k X_{\alpha_p,i_p,j_p} A(YA)^l \\
&= \sum_{\substack{k+l=0 \\ k\geq 0,\, l\geq 0}}^{n-2} A(YA)^k X_{\alpha_p,i_p,j_p} A(YA)^{k+l-k} \\
&= \sum_{k=0}^{n-2} \sum_{l=0}^{k} A(YA)^l X_{\alpha_p,i_p,j_p} A(YA)^{k-l} \\
&= \sum_{k=1}^{n-1} \sum_{l=0}^{k-1} A(YA)^l X_{\alpha_p,i_p,j_p} A(YA)^{k-l-1}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\Psi_{\alpha_p,i_p,j_p} \circ \Psi_Y(A) &= \sum_{k=0}^{n-1} A(YA)^k + \sum_{k=0}^{n-1} + \sum_{l=0}^{k-1} A(YA)^l X_{\alpha_p,i_p,j_p} A(YA)^{k-l-1} \\
&= A + \sum_{k=1}^{n-1} \left[A(YA)^k + \sum_{l=0}^{k-1} A(YA)^l X_{\alpha_p,i_p,j_p} A(YA)^{k-l-1} \right] \\
&= A + \sum_{k=1}^{n-1} A((Y + X_{\alpha_p,i_p,j_p})A)^k \\
&= A + \sum_{k=1}^{n-1} A(XA)^k \\
&= A + \sum_{k=1}^{n-2} A(XA)^k = \Psi_X(A)
\end{aligned}
$$

for all $A \in \mathcal{PAT}_n(\mathbb{F})$, where use has been made of the identity that for $k \geq 1$, Thank you very much!