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ICM 2014 Satellite

19TH CONFERENCE OF THE INTERNATIONAL LINEAR ALGEBRA SOCIETY

Solidarity in Linear Algebra

SEOUL, KOREA
Sungkyunkwan University
August 6-9, 2014

PLENARY SPEAKERS

Ravindra Bapat (LAMA Lecturer)
Peter Benner
Dario Bini (LAA Lecturer)
Shaun Fallat (Taussky-Todd Lecturer)
Andreas Frommer (SIAG/LA Lecturer)
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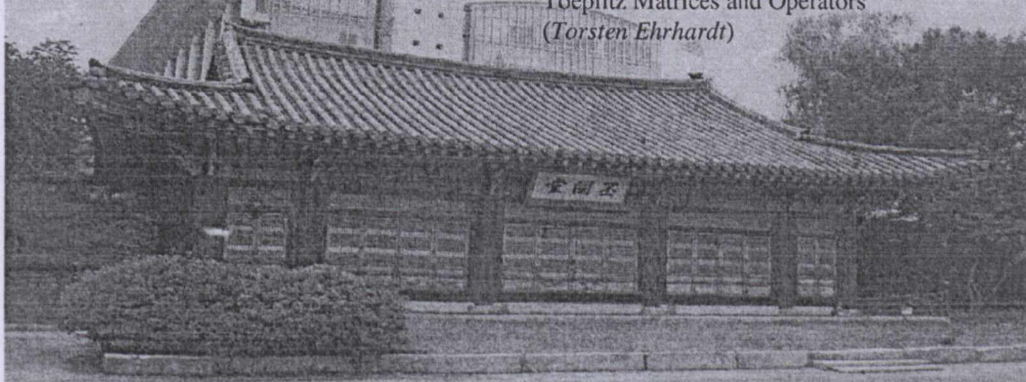

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Conference Theme

Solidarity in Linear Algebra

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- **Peter Benner** (Chemnitz Univ. of Technology, Germany)
- **Dario Bini** (University of Pisa, Italy)/ LAA Lecturer)
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LINEAR SPACES AND PRESERVERS OF BOUNDED RANK-TWO PER-SYMMETRIC TRIANGULAR MATRICES

W.L. CHOOI*, K.H. KWA*, M.H. LIM*, AND Z.C. NG†

Abstract. Let $\mathbb{F}$ be a field and m, n be integers $m, n \geqslant 3$. Let $\mathcal{SM}_n(\mathbb{F})$ and $\mathcal{ST}_n(\mathbb{F})$ denote the linear space of $n \times n$ per-symmetric matrices over $\mathbb{F}$ and the linear space of $n \times n$ per-symmetric triangular matrices over $\mathbb{F}$, respectively. In this talk, the structure of linear subspaces of bounded rank-two matrices of $\mathcal{ST}_n(\mathbb{F})$ will be given. Using this structural result, a classification of bounded rank-two linear preservers $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_m(\mathbb{F})$, with $\mathbb{F}$ of characteristic not two, is obtained. As a corollary, a complete description of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two is addressed.

Key words. Per-symmetric triangular matrices, Rank, Spaces of bounded rank-two matrices, Bounded rank-two linear preservers

AMS subject classifications. 15A03, 15A04, 15A86.

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Linear spaces and preservers of bounded rank two per-symmetric triangular matrices

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In this talk, we will give

- the structure of linear spaces of bounded rank two per-symmetric triangular matrices, and
- a classification of bounded rank two linear preservers on per-symmetric triangular matrices over a field of characteristic not two.

Motivation

Let $\mathbb{F}$ be a field and m, n be positive integers. Given an $m \times n$ matrix $A = (a_{ij}) \in \mathcal{M}_{m,n}(\mathbb{F})$, let $A^+ = (b_{ij}) \in \mathcal{M}_{n,m}(\mathbb{F})$ be such that $b_{ij} = a_{m+1-j,n+1-i}$ for all i, j.

An n-square upper triangular matrix $A \in \mathcal{T}_n(\mathbb{F})$ is said to be *per-symmetric* if it is symmetric around the minor diagonal, i.e., $A^+ = A$. We use $\mathcal{ST}_n(\mathbb{F})$ to denote the linear space of all n-square per-symmetric triangular matrices over $\mathbb{F}$.

Note that $E_{i,n+1-i}$ with $1 \leqslant i \leqslant \frac{n+1}{2}$, and $E_{i,j} + E_{i,j}^+$ with $1 \leqslant i \leqslant j < n+1-i$, are in $\mathcal{ST}_n(\mathbb{F})$, and these elements form the standard basis of $\mathcal{ST}_n(\mathbb{F})$.

Motivation

Let us begin with an observation.

Recall that any rank k symmetric $A \in \mathcal{M}_n(\mathbb{F})$ over a field $\mathbb{F}$ of characteristic not two can be decomposited as

$$A = P\left(\sum_{i=1}^{k} \alpha_i E_{i,i}\right) P^T .$$

Here, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible and α_i's are nonzero scalars in $\mathbb{F}$.

Motivation

We now have the decomposition of per-symmetric triangular matrices. To see this, we first denote

$$Z_{i,j}^{\alpha} := E_{i,j} + E_{i,j}^{+} + \alpha E_{i,n+1-i}$$

for $\alpha \in \mathbb{F}$ and i, j are integers satisfying $1 \leqslant i, j \leqslant n$ with $j \neq n+1-i$. Evidently, $Z_{i,j}^{\alpha}$ is of rank two, and $Z_{i,j}^{\alpha} \in \mathcal{ST}_n(\mathbb{F})$ whenever $1 \leqslant i \leqslant j < n+1-i$.

Motivation

Theorem 1

Let $\mathbb{F}$ be a field and n be an integer $n \geqslant 2$. Then $A \in \mathcal{T}_n(\mathbb{F})$ is a rank k per-symmetric matrix if and only if there exist an invertible matrix $P \in \mathcal{T}_n(\mathbb{F})$, an integer $0 \leqslant h \leqslant \frac{k}{2}$, scalars $\alpha_1, \ldots, \alpha_h \in \mathbb{F}$ and nonzero scalars $\beta_{2h+1}, \ldots, \beta_k \in \mathbb{F}$ such that

$$A = P \left(\sum_{i=1}^{h} Z_{s_i,t_j}^{\alpha_i} + \sum_{j=2h+1}^{k} \beta_j E_{p_j,n+1-p_j} \right) P^{+}$$

where $\{s_1, n+1-t_1, \ldots, s_h, n+1-t_h, p_{2h+1}, \ldots, p_k\}$ and $\{t_1, n+1-s_1, \ldots, t_h, n+1-s_h, n+1-p_{2h+1}, \ldots, n+1-p_k\}$ are two sets of k distinct integers such that $1 \leqslant s_i \leqslant t_i < n+1-s_i$ for $i = 1, \ldots, h$, and $1 \leqslant p_i \leqslant \frac{n+1}{2}$ for $i = 2h+1, \ldots, k$; and $(\alpha_1, \ldots, \alpha_k) \neq 0$ only if $\operatorname{char} \mathbb{F} = 2$.

Motivation

In particular, if $A \in \mathcal{ST}_n(\mathbb{F})$ is of rank two, then there exists an invertible matrix $P \in \mathcal{T}_n(\mathbb{F})$ such that either

$$A = P(\alpha E_{i,n+1-i} + \beta E_{j,n+1-j})P^+$$

for some nonzero scalars $\alpha, \beta \in \mathbb{F}$ and integers $1 \leqslant i \neq j \leqslant \frac{n+1}{2}$; or

$$A = PZ_{ij}^{\lambda}P^+$$

for some integers $1 \leqslant i \leqslant j < n+1-i$ and scalar $\lambda \in \mathbb{F}$ with $\lambda \neq 0$ only if $\operatorname{char}\mathbb{F} = 2$.

Definitions

Inspired by this observation, we define

$$u \oslash v := u \cdot v^+ + v \cdot u^+ \qquad \text{and} \qquad u^2 := u \cdot u^+$$

for every $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$, where $u \cdot v^+$ is the usual matrix product of $u \in \mathcal{M}_{n,1}(\mathbb{F})$ and $v^+ \in \mathbb{F}^n$. We see that $(u, v) \mapsto u \oslash v$ is a symmetric bilinear map from $\mathcal{M}_{n,1}(\mathbb{F}) \times \mathcal{M}_{n,1}(\mathbb{F})$ into $\mathcal{M}_n(\mathbb{F})$, and

$$e_i \oslash e_j = E_{i,n+1-j} + E^+_{i,n+1-j} \quad \text{and} \quad e_i^2 = E_{i,n+1-i},$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathcal{M}_{n,1}(\mathbb{F})$.

It can be verified that $u \oslash v = 0$ if and only if $u = 0$ or $v = 0$ when $\operatorname{char} \mathbb{F} \neq 2$, whereas u, v are linearly dependent when $\operatorname{char} \mathbb{F} = 2$. Also, $\operatorname{rank}(u \oslash v) = 2$ if and only if u, v are linearly independent; and $\operatorname{rank}(a(u \oslash v) + bu^2 + cu^2) \leqslant 2$ for every $a, b, c \in \mathbb{F}$.

Definitions

Let $1 \leqslant i \leqslant n$. Denote

$$\mathcal{U}_{i,n} := \{(x_1, \ldots, x_i, 0, \ldots, 0)^T \in \mathcal{M}_{n,1}(\mathbb{F}) : x_1, \ldots, x_i \in \mathbb{F}\}.$$

When n is clear from the context, $\mathcal{U}_{i,n}$ is abbreviated to $\mathcal{U}_i$.

A matrix $A \in \mathcal{ST}_n(\mathbb{F})$ is of rank at most two if and only if

$$A = \alpha u^2 + \beta v^2$$

for some linearly independent $u, v \in \mathcal{U}_p$ with $1 \leqslant p \leqslant \frac{n+1}{2}$ and $\alpha, \beta \in \mathbb{F}$; or

$$A = u \oslash v + \lambda u^2$$

for some linearly independent $u \in \mathcal{U}_p$, $v \in \mathcal{U}_q$ with $1 \leqslant p \leqslant q < n + 1 - p$ and $\lambda \in \mathbb{F}$, with $\lambda \neq 0$ only if $\operatorname{char} \mathbb{F} = 2$.

Linear spaces of bounded rank two

A linear subspace $\mathcal{S}$ of $\mathcal{ST}_n(\mathbb{F})$ is said to be *bounded rank two* if $\mathcal{S}$ consists of matrices of rank at most two.

Let $u \in \mathcal{M}_{n,1}(\mathbb{F})$ and $\mathcal{V}$ be a linear subspace of $\mathcal{M}_{n,1}(\mathbb{F})$. We denote

$$u \oslash \mathcal{V} := \{u \oslash v : v \in \mathcal{V}\}.$$

It is immediate that $u \oslash \mathcal{V}$ is a linear space of bounded rank two.

We now give a classification of the structure of linear spaces of bounded rank two per-symmetric triangular matrices over an arbitrary field.

Linear spaces of bounded rank two

Theorem 2

Let $\mathbb{F}$ be a field with at least three elements and n be an integer $n \geqslant 2$. Let $\mathcal{S}$ be a linear subspace of $\mathcal{ST}_n(\mathbb{F})$. Then $\mathcal{S}$ is bounded rank two if and only if one of the following holds:

(a) $\mathcal{S} \subseteq \left\langle u^2, v^2, u \oslash v \right\rangle$ *for some linearly independent* $u, v \in \mathcal{U}_p$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$.

(b) $\mathcal{S} = u \oslash \mathcal{V}$ *for some nonzero* $u \in \mathcal{U}_p$ *and some linear subspace* $\mathcal{V}$ *of* $\mathcal{U}_q$ *with* $1 \leqslant p \leqslant n+1-q \leqslant n$.

(c) $\mathcal{S} = u \oslash \mathcal{V} + \langle u^2 \rangle$ *for some nonzero* $u \in \mathcal{U}_p$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and some linear subspace* $\mathcal{V}$ *of* $\mathcal{U}_q$ *with* $1 \leqslant q \leqslant n+1-p \leqslant n$; *and* $\mathcal{S}$ *is of this form only if* $\operatorname{char} \mathbb{F} = 2$.

Linear spaces of bounded rank two

Theorem 2

(d) $\mathcal{S} = \left\langle u \oslash v_1 + \lambda_1 u^2, \ldots, u \oslash v_k + \lambda_k u^2 \right\rangle$ *for some scalars* $\lambda_1, \ldots, \lambda_k \in \mathbb{F}$ *with* $(\lambda_1, \ldots, \lambda_k) \neq 0$*, and some linearly independent vectors* $u, v_1, \ldots, v_k$ *such that* $u \in \mathcal{U}_p$*,* $v_1, \ldots, v_k \in \mathcal{U}_q$ *with* $1 \leqslant p \leqslant \frac{n+1}{2}$ *and* $1 \leqslant q \leqslant n+1-p \leqslant n$*; and* $\mathcal{S}$ *is of this form only if* $\mathit{char}\,\mathbb{F} = 2$.

(e) $\mathcal{S} = \langle u \oslash v, u \oslash w, v \oslash w \rangle$ *for some linearly independent vectors* $u \in \mathcal{U}_p$*,* $v \in \mathcal{U}_q$*,* $w \in \mathcal{U}_r$ *such that* $1 \leqslant p, q \leqslant n+1-r$ *and* $p \leqslant n+1-q$*; and* $\mathcal{S}$ *is of this form only if* $\mathit{char}\,\mathbb{F} = 2$.

Linear spaces of bounded rank two - a remark

Suppose that $\mathbb{F}$ is a field of characteristic not two.
By the observation of

$$u \oslash v + \alpha u^2 = u \oslash (v + \frac{\alpha}{2} u)$$

for every $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ and $\alpha \in \mathbb{F}$, we notice that any linear space of bounded rank two of Form (c) of (d) in Theorem 2 can be simplified to Form (b) in Theorem 1.

Also, a linear space of bounded rank two of Form (e) contains rank three matrices. For example

$$\operatorname{rank}(u \ominus v + v \ominus w + w \ominus u) = 3$$

whenever u, v, w are linearly independent.

Bounded rank two linear preservers

A linear mapping $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$ is said to be a *bounded rank two linear preserver* if

$$1 \leqslant \operatorname{rank} \psi(A) \leqslant 2 \quad \text{whenever} \quad 1 \leqslant \operatorname{rank} A \leqslant 2,$$

We now give a classification of bounded rank two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two.

Bounded rank two linear preservers

Theorem 3

Let $\mathbb{F}$ be a field of characteristic not two and m, n be integers such that $m, n \geqslant 3$. Then $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{ST}_m(\mathbb{F})$ is a bounded rank two linear preserver if and only if $m \geqslant n$ and ψ is of one of the following forms:

(i) *There exist a nonzero vector $u \in \mathcal{U}_{p,m}$ and a linear mapping $\varphi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{U}_{q,m}$, with $1 \leqslant p \leqslant m + 1 - q$, such that*

$$\psi(A) = u \oslash \varphi(A) \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where $\varphi(A) \neq 0$ for every nonzero bounded rank two matrix $A \in \mathcal{ST}_n(\mathbb{F})$.

Bounded rank two linear preservers

Theorem 3

(ii) *There exist a full rank matrix $P \in \mathcal{M}_{m,n}(\mathbb{F})$ and a nonzero $\lambda \in \mathbb{F}$ such that*

$$\psi(A) = \lambda P A P^{+} \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where $Pe_i \in \mathcal{U}_{p_i,m} \setminus \mathcal{U}_{p_i-1,m}$ for $i = 1, \ldots, n$ such that $1 \leqslant p_i \leqslant \frac{m+1}{2}$ for every $1 \leqslant i \leqslant \frac{n+1}{2}$, and $p_i \leqslant m + 1 - p_j$ for every $1 \leqslant i < j \leqslant n + 1 - i$. In particular, $P \in \mathcal{T}_n(\mathbb{F})$ when $m = n$.

Bounded rank two linear preservers

Theorem 3

(iii) *When $n = 4$, in addition to (i) and (ii), ψ also takes the form*

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta)a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^{+}$$

for all $A = (a_{ij}) \in \mathcal{ST}_4(\mathbb{F})$, where $\alpha, \beta, \theta \in \mathbb{F}$ are scalars such that α, β are nonzero with $\beta \neq 2\alpha$, and θ is nonzero only if $|\mathbb{F}| = 3$, and $P \in \mathcal{M}_{m,4}(\mathbb{F})$ is a full rank matrix in which $Pe_i \in \mathcal{U}_{p_i,m}$ for $1 \leqslant i \leqslant 4$ with $1 \leqslant p_i \leqslant \frac{m+1}{2}$ for every $1 \leqslant i \leqslant 2$, and $p_i \leqslant m + 1 - p_j$ for every $1 \leqslant i < j \leqslant 5 - i$. In particular, $P \in \mathcal{T}_4(\mathbb{F})$ when $m = 4$.

Bounded rank two linear preservers

Theorem 3

(iv) *When $n = 3$, in addition to* (i) *and* (ii), ψ *also takes one of the following forms:*

(iv)(a) *There exist a surjective linear mapping $\phi : \mathcal{ST}_3(\mathbb{F}) \to \mathbb{F}^3$ and a full rank matrix $P \in \mathcal{M}_{m,2}(\mathbb{F})$ such that*

$$\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^+$$

for all $A \in \mathcal{ST}_4(\mathbb{F})$, where $Pe_1, Pe_2 \in \mathcal{U}_{p,m}$ for some integer $1 \leqslant p \leqslant \frac{m+1}{2}$, $\phi(A)_i$ is the i-th component of $\phi(A) \in \mathbb{F}^3$, and $\phi(A) \neq 0$ for every nonzero bounded rank-two matrix $A \in \mathcal{ST}_3(\mathbb{F})$.

Bounded rank two linear preservers

Theorem 3

(iv)(b) *There exist scalars* $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ *with* $\lambda_3 \neq 0$ *such that either*

$$\psi(A) = P \begin{bmatrix} a_{pp} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} & \eta_1 a_{12} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, *where* $\eta_1, \eta_2 \in \mathbb{F}$ *are nonzero and* $\{p, q\} = \{1, 2\}$; *or*

$$\psi(A) = P \begin{bmatrix} a_{pp} & a_{1s} + \lambda_2 a_{qq} & \eta_1 a_{1t} + \lambda_1 a_{qq} \\ 0 & \lambda_3 a_{qq} & a_{1s} + \lambda_2 a_{qq} \\ 0 & 0 & a_{pp} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in \mathcal{ST}_3(\mathbb{F})$, *where* $\eta \in \mathbb{F}$ *is nonzero and* $\{p, q\} = \{s, t\} = \{1, 2\}$. *Here,* $P \in \mathcal{M}_{m,3}(\mathbb{F})$ *is a full rank matrix such that* $Pe_1, Pe_2 \in \mathcal{U}_{p,m}$ *with* $1 \leqslant p \leqslant \frac{m+1}{2}$ *and* $Pe_3 \in \mathcal{U}_{q,m}$ *with* $1 \leqslant q \leqslant m + 1 - p$. *In particular,* $P \in \mathcal{T}_3(\mathbb{F})$ *when* $m = 3$.

Why not rank one linear preservers on per-symmetric triangular matrices ?

Let $\mathbb{F}$ be a field and m, n be integers $\geqslant 2$. Let $p := \left\lfloor \frac{n+1}{2} \right\rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Let $\psi : \mathcal{ST}_n(\mathbb{F}) \to \mathcal{SM}_m(\mathbb{F})$ be the linear mapping defined by

$$\psi(A) = \lambda P \begin{bmatrix} \phi(A_1) & \varphi(A_2) \\ 0 & \phi(A_1)^+ \end{bmatrix} P^+ \quad \text{for } A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1^+ \end{bmatrix} \in \mathcal{ST}_n(\mathbb{F})$$

with $A_1 \in \mathcal{T}_{p,n-p}(\mathbb{F})$ and $A_2 \in \mathcal{SM}_p(\mathbb{F})$, where $\lambda \in \mathbb{F}\backslash\{0\}$, $P \in \mathcal{M}_{m,n}(\mathbb{F})$ is of full rank, and $\phi : \mathcal{T}_{p,n-p}(\mathbb{F}) \to \mathcal{M}_{p,n-p}(\mathbb{F})$ and $\varphi : \mathcal{SM}_p(\mathbb{F}) \to \mathcal{SM}_p(\mathbb{F})$ are linear mappings. Here $\mathcal{T}_{p,n-p}(\mathbb{F}) = \mathcal{T}_p(\mathbb{F})$ when $n - p = p$, and

$$\mathcal{T}_{p,n-p}(\mathbb{F}) = \left\{ \begin{bmatrix} T \\ 0 \end{bmatrix} \in \mathcal{M}_{p,n-p}(\mathbb{F}) \;\middle|\; T \in \mathcal{T}_{p-1}(\mathbb{F}) \right\} \text{ when } n-p = p-1.$$

It can be verified that

- ψ is a rank one linear preserver whenever φ is a rank one linear preserver on $\mathcal{SM}_p(\mathbb{F})$, and
- ψ is rank one non-increasing whenever φ is a rank one non-increasing linear mapping on $\mathcal{SM}_p(\mathbb{F})$.

By the structural results of rank-one linear preservers and rank one non-increasing linear mappings on symmetric matrices, the structure of ψ can be established immediately.

Thank You