19TH CONFERENCE OF THE INTERNATIONAL LINEAR ALGEBRA SOCIETY

SEOUL, KOREA
Sungkyunkwan University
August 6-9, 2014

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Peter Benner
Dario Bini (LAA Lecturer)
Shaun Fallat (Taussky-Todd Lecturer)
Andreas Frommer (SIAG/LA Lecturer)
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Combinatorial Problems in Linear Algebra (Richard A. Brualdi and Geir Dahl)
Matrix Inequalities (Fuzhen Zhang and Minghua Lin)
Spectral Theory of Graphs and Hypergraphs (Vladimir S. Nikiforov)
Tensor Eigenvalues (Jia-Yu Shao and Lifun Qi)
Quantum Information and Computing (Chi-Kwong Li and Yiu Tung Poon)
 Riordan arrays and Related Topics (Gi-Sang Cheon and Louis W. Shapiro)
Nonnegative Matrices and Generalizations (Judy McDonald)
Tropical Matrices and Operators (Torsten Ehrhardt)

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Conference Theme
Solidarity in Linear Algebra

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Plenary Speakers
- Ravi Bapat (Indian Statistical Institute, India)
- Peter Benner (Chemnitz Univ. of Technology, Germany)
- Dario Bini (University of Pisa, Italy)/ LAA Lecturer
- Ljiljana Cvetkovic (University of Novi Sad, Serbia)
- Shaun Fallat (University of Regina, Canada)/ Taussky-Todd Lecture
- Andreas Frommer (University of Wuppertal, Germany- SIAG/LA)
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LINEAR SPACES AND PRESERVERS OF BOUNDED RANK-TWO PER-SYMMETRIC TRIANGULAR MATRICES

W.L. CHOOI*, K.H. KWA*, M.H. LIM*, AND Z.C. NG1

Abstract. Let $\mathbb{F}$ be a field and $m, n$ be integers $m, n \geq 3$. Let $\mathcal{S}M_n(\mathbb{F})$ and $\mathcal{S}T_n(\mathbb{F})$ denote the linear space of $n \times n$ per-symmetric matrices over $\mathbb{F}$ and the linear space of $n \times n$ per-symmetric triangular matrices over $\mathbb{F}$, respectively. In this talk, the structure of linear subspaces of bounded rank-two matrices of $\mathcal{S}T_n(\mathbb{F})$ will be given. Using this structural result, a classification of bounded rank-two linear preservers $\psi: \mathcal{S}T_n(\mathbb{F}) \rightarrow \mathcal{S}M_m(\mathbb{F})$, with $\mathbb{F}$ of characteristic not two, is obtained. As a corollary, a complete description of bounded rank-two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two is addressed.

Key words. Per-symmetric triangular matrices, Rank, Spaces of bounded rank-two matrices, Bounded rank-two linear preservers

AMS subject classifications. 15A03, 15A04, 15A86.

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Linear spaces and preservers of bounded rank two per-symmetric triangular matrices

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The 19th Conference of the
International Linear Algebra Society 2014
Sungkyunkwan University, Seoul, Korea
August 6-9, 2014

a joint work with K.H. Kwa, M.H. Lim and S.Z. Ng
Let $F$ be a field and $m, n$ be positive integers. Given an $m \times n$ matrix $A = (a_{ij}) \in M_{m,n}(F)$, let $A^+ = (b_{ij}) \in M_{n,m}(F)$ be such that $A$ and $A^+$ are per-symmetric triangular matrices.

An $n \times n$ upper triangular matrix $A \in M_n(F)$ is said to be per-symmetric if $A + A^T$ is symmetric. Let $ST_n(F)$ denote the linear space of all $n \times n$ square per-symmetric triangular matrices over $F$.

Note that $E_{i,i}$ with $1 \leq i \leq \frac{n+1}{2}$, and $E_{ij} + E_{ji}$ with $1 \leq i < j < n + 1 - i$, are in $ST_n(F)$, and these elements form the standard basis of $ST_n(F)$.

In this talk, we will give

- the structure of linear spaces of bounded rank two per-symmetric triangular matrices, and
- a classification of bounded rank two linear preservers on per-symmetric triangular matrices over a field of characteristic not two.

W. L. Chooi

Linear spaces and preservers of bounded rank two
Let $\mathbb{F}$ be a field and $m, n$ be positive integers. Given an $m \times n$ matrix $A = (a_{ij}) \in M_{m,n}(\mathbb{F})$, let $A^+ = (b_{ij}) \in M_{n,m}(\mathbb{F})$ be such that $b_{ij} = a_{m+1-j,n+1-i}$ for all $i, j$.

An $n$-square upper triangular matrix $A \in T_n(\mathbb{F})$ is said to be per-symmetric if it is symmetric around the minor diagonal, i.e., $A^+ = A$. We use $ST_n(\mathbb{F})$ to denote the linear space of all $n$-square per-symmetric triangular matrices over $\mathbb{F}$.

Note that $E_{i,n+1-i}$ with $1 \leq i \leq \frac{n+1}{2}$, and $E_{i,j} + E_{i,j}^+$ with $1 \leq i \leq j < n + 1 - i$, are in $ST_n(\mathbb{F})$, and these elements form the standard basis of $ST_n(\mathbb{F})$. 

Motivation

Let us begin with an observation.

Recall that any rank $k$ symmetric $A \in \mathcal{M}_n(\mathbb{F})$ over a field $\mathbb{F}$ of characteristic not two can be decomposed as

$$A = P \left( \sum_{i=1}^{k} \alpha_i E_{i,i} \right) P^T.$$ 

Here, $P \in \mathcal{M}_n(\mathbb{F})$ is invertible and $\alpha_i$'s are nonzero scalars in $\mathbb{F}$. 
We now have the decomposition of per-symmetric triangular matrices. To see this, we first denote

\[ Z_{i,j}^{\alpha} := E_{i,j} + E_{i,j}^+ + \alpha E_{i,n+1-i} \]

for \( \alpha \in \mathbb{F} \) and \( i, j \) are integers satisfying \( 1 \leq i, j \leq n \) with \( j \neq n + 1 - i \). Evidently, \( Z_{i,j}^{\alpha} \) is of rank two, and \( Z_{i,j}^{\alpha} \in ST_n(\mathbb{F}) \) whenever \( 1 \leq i \leq j < n + 1 - i \).
Motivation

**Theorem 1**

Let $\mathbb{F}$ be a field and $n$ be an integer $n \geq 2$. Then $A \in \mathcal{T}_n(\mathbb{F})$ is a rank $k$ per-symmetric matrix if and only if there exist an invertible matrix $P \in \mathcal{T}_n(\mathbb{F})$, an integer $0 \leq h \leq \frac{k}{2}$, scalars $\alpha_1, \ldots, \alpha_h \in \mathbb{F}$ and nonzero scalars $\beta_{2h+1}, \ldots, \beta_k \in \mathbb{F}$ such that

$$A = P \left( \sum_{i=1}^{h} Z_{s_i,t_i}^{\alpha_i} + \sum_{j=2h+1}^{k} \beta_j E_{p_j,n+1-p_j} \right) P^+$$

where $\{s_1, n+1-t_1, \ldots, s_h, n+1-t_h, p_{2h+1}, \ldots, p_k\}$ and $\{t_1, n+1-s_1, \ldots, t_h, n+1-s_h, n+1-p_{2h+1}, \ldots, n+1-p_k\}$ are two sets of $k$ distinct integers such that $1 \leq s_i \leq t_i < n+1-s_i$ for $i = 1, \ldots, h$, and $1 \leq p_i \leq \frac{n+1}{2}$ for $i = 2h+1, \ldots, k$; and $(\alpha_1, \ldots, \alpha_k) \neq 0$ only if $\text{char} \mathbb{F} = 2$. 
Motivation

In particular, if \( A \in ST_n(\mathbb{F}) \) is of rank two, then there exists an invertible matrix \( P \in \mathcal{T}_n(\mathbb{F}) \) such that either

\[
A = P(\alpha E_{i,n+1-i} + \beta E_{j,n+1-j})P^+
\]

for some nonzero scalars \( \alpha, \beta \in \mathbb{F} \) and integers \( 1 \leq i \neq j \leq \frac{n+1}{2} \); or

\[
A = PZ_{ij}^\lambda P^+
\]

for some integers \( 1 \leq i \leq j < n+1 - i \) and scalar \( \lambda \in \mathbb{F} \) with \( \lambda \neq 0 \) only if \( \text{char} \mathbb{F} = 2 \).
Definitions

Inspired by this observation, we define

\[ u \otimes v := u \cdot v^+ + v \cdot u^+ \quad \text{and} \quad u^2 := u \cdot u^+ \]

for every \( u, v \in \mathcal{M}_{n,1}(\mathbb{F}) \), where \( u \cdot v^+ \) is the usual matrix product of \( u \in \mathcal{M}_{n,1}(\mathbb{F}) \) and \( v^+ \in \mathbb{F}^n \). We see that \( (u, v) \mapsto u \otimes v \) is a symmetric bilinear map from \( \mathcal{M}_{n,1}(\mathbb{F}) \times \mathcal{M}_{n,1}(\mathbb{F}) \) into \( \mathcal{M}_n(\mathbb{F}) \), and

\[ e_i \otimes e_j = E_{i,n+1-j}^+ + E_{i,n+1-j}^+ \quad \text{and} \quad e_j^2 = E_{i,n+1-j}^+ , \]

where \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathcal{M}_{n,1}(\mathbb{F}) \).

It can be verified that \( u \otimes v = 0 \) if and only if \( u = 0 \) or \( v = 0 \) when \( \text{char } \mathbb{F} \neq 2 \), whereas \( u, v \) are linearly dependent when \( \text{char } \mathbb{F} = 2 \). Also, \( \text{rank} (u \otimes v) = 2 \) if and only if \( u, v \) are linearly independent; and \( \text{rank} (a(u \otimes v) + bu^2 + cu^2) \leq 2 \) for every \( a, b, c \in \mathbb{F} \).
Let $1 \leq i \leq n$. Denote

$$U_{i,n} := \{(x_1, \ldots, x_i, 0, \ldots, 0)^T \in M_{n,1}(F) : x_1, \ldots, x_i \in F\}.$$ 

When $n$ is clear from the context, $U_{i,n}$ is abbreviated to $U_i$.

A matrix $A \in ST_n(F)$ is of rank at most two if and only if

$$A = \alpha u^2 + \beta v^2$$

for some linearly independent $u, v \in U_p$ with $1 \leq p \leq \frac{n+1}{2}$ and $\alpha, \beta \in F$; or

$$A = u \otimes v + \lambda u^2$$

for some linearly independent $u \in U_p$, $v \in U_q$ with $1 \leq p \leq q < n + 1 - p$ and $\lambda \in F$, with $\lambda \neq 0$ only if char $F = 2$. 
Linear spaces of bounded rank two

A linear subspace $S$ of $ST_n(\mathbb{F})$ is said to be bounded rank two if $S$ consists of matrices of rank at most two.

Let $u \in \mathcal{M}_{n,1}(\mathbb{F})$ and $\mathcal{V}$ be a linear subspace of $\mathcal{M}_{n,1}(\mathbb{F})$. We denote

$$u \odot \mathcal{V} := \{ u \odot v : v \in \mathcal{V} \}.$$ 

It is immediate that $u \odot \mathcal{V}$ is a linear space of bounded rank two.

We now give a classification of the structure of linear spaces of bounded rank two per-symmetric triangular matrices over an arbitrary field.
Let $\mathbb{F}$ be a field with at least three elements and $n$ be an integer $n \geq 2$. Let $S$ be a linear subspace of $\text{ST}_n^2(\mathbb{F})$. Then $S$ is bounded rank two if and only if one of the following holds:

(a) $S \subseteq \langle u^2, v^2, u \circ v \rangle$ for some linearly independent $u, v \in \mathcal{U}_p$ with $1 \leq p \leq \frac{n+1}{2}$.

(b) $S = \langle u \circ v \rangle$ for some nonzero $u \in \mathcal{U}_p$ and some linear subspace $\gamma$ of $\mathcal{U}_q$ with $1 \leq p \leq n + 1 - q \leq n$.

(c) $S = \langle u^2, \gamma \rangle$ for some nonzero $u \in \mathcal{U}_p$ with $1 \leq p \leq \frac{n+1}{2}$ and some linear subspace $\gamma$ of $\mathcal{U}_q$ with $1 \leq q \leq n + 1 - p \leq n$, and $S$ is of this form only if $\text{char} \mathbb{F} = 2$. 

Theorem 2
(d) \[ S = \left\{ u \circ v, \ u \circ w, \ v \circ w \right\} \] for some linearly independent vectors \( u, v, w \in U_q \), such that \( 1 \leq p \leq n+1-r \) and \( q \leq n+1-q \); and \( S \) is of this form only if \( \text{char } \mathbb{F} = 2 \).

(e) \[ S = \left\{ u \circ v, \ u \circ w, \ v \circ w \right\} \] for some linearly independent vectors \( u, v, w \in U_p \), such that \( 1 \leq p \leq n+1-r \) and \( q \leq n+1-q \); and \( S \) is of this form only if \( \text{char } \mathbb{F} = 2 \).
Linear spaces of bounded rank two - a remark

Suppose that $\mathbb{F}$ is a field of characteristic not two. By the observation of

$$u \odot v + \alpha u^2 = u \odot (v + \frac{\alpha}{2} u)$$

for every $u, v \in \mathcal{M}_{n,1}(\mathbb{F})$ and $\alpha \in \mathbb{F}$, we notice that any linear space of bounded rank two of Form (c) of (d) in Theorem 2 can be simplified to Form (b) in Theorem 1.

Also, a linear space of bounded rank two of Form (e) contains rank three matrices. For example

$$\text{rank} \left( u \odot v + v \odot w + w \odot u \right) = 3$$

whenever $u, v, w$ are linearly independent.
A linear mapping \( \psi : ST_n(\mathbb{F}) \to ST_m(\mathbb{F}) \) is said to be a bounded rank two linear preserver if

\[
1 \leq \text{rank} \, \psi(A) \leq 2 \quad \text{whenever} \quad 1 \leq \text{rank} \, A \leq 2,
\]

We now give a classification of bounded rank two linear preservers between per-symmetric triangular matrix spaces over a field of characteristic not two.
Theorem 3

Let $\mathbb{F}$ be a field of characteristic not two and $m, n$ be integers such that $m, n \geq 3$. Then $\psi : ST_n(\mathbb{F}) \to ST_m(\mathbb{F})$ is a bounded rank two linear preserver if and only if $m \geq n$ and $\psi$ is of one of the following forms:

(i) There exist a nonzero vector $u \in U_{p,m}$ and a linear mapping $\varphi : ST_n(\mathbb{F}) \to U_{q,m}$, with $1 \leq p \leq m + 1 - q$, such that

$$\psi(A) = u \circ \varphi(A) \quad \text{for all} \ A \in ST_n(\mathbb{F}),$$

where $\varphi(A) \neq 0$ for every nonzero bounded rank two matrix $A \in ST_n(\mathbb{F})$. 
Theorem 3

(ii) There exist a full rank matrix $P \in \mathcal{M}_{m,n}(\mathbb{F})$ and a nonzero $\lambda \in \mathbb{F}$ such that

$$\psi(A) = \lambda P A P^+ \quad \text{for all } A \in \mathcal{ST}_n(\mathbb{F}),$$

where $P e_i \in \mathcal{U}_{p_i,m} \setminus \mathcal{U}_{p_i-1,m}$ for $i = 1, \ldots, n$ such that

$1 \leq p_i \leq \frac{m+1}{2}$ for every $1 \leq i \leq \frac{n+1}{2}$, and

$p_i \leq m + 1 - p_j$ for every $1 \leq i < j \leq n + 1 - i$. In particular, $P \in \mathcal{T}_n(\mathbb{F})$ when $m = n$. 
Bounded rank two linear preservers

Theorem 3

(iii) When $n = 4$, in addition to (i) and (ii), $\psi$ also takes the form

$$\psi(A) = P \begin{bmatrix} a_{11} & a_{12} & \alpha a_{13} + \theta(a_{14} - a_{23}) & \beta a_{14} \\ 0 & a_{22} & (2\alpha - \beta) a_{23} & \alpha a_{13} + \theta(a_{14} - a_{23}) \\ 0 & 0 & a_{22} & a_{12} \\ 0 & 0 & 0 & a_{11} \end{bmatrix} P^+$$

for all $A = (a_{ij}) \in ST_4(\mathbb{F})$, where $\alpha, \beta, \theta \in \mathbb{F}$ are scalars such that $\alpha, \beta$ are nonzero with $\beta \neq 2\alpha$, and $\theta$ is nonzero only if $|\mathbb{F}| = 3$, and $P \in \mathcal{M}_{m,4}(\mathbb{F})$ is a full rank matrix in which $Pe_i \in \mathcal{U}_{p_i,m}$ for $1 \leq i \leq 4$ with $1 \leq p_i \leq \frac{m+1}{2}$ for every $1 \leq i \leq 2$, and $p_i \leq m + 1 - p_j$ for every $1 \leq i < j \leq 5 - i$. In particular, $P \in \mathcal{I}_4(\mathbb{F})$ when $m = 4$. 
Theorem 3.

(iv) When \( n = 3 \), in addition to (i) and (ii), \( \psi \) also takes one of the following forms:

(iv)(a) There exist a surjective linear mapping \( \phi : \mathcal{S} \mathcal{T}_3(F) \to F^3 \) and a full rank matrix \( P \in \mathcal{M}_{m,2}(F) \) such that

\[
\psi(A) = P \begin{bmatrix} \phi(A)_3 & \phi(A)_1 \\ \phi(A)_2 & \phi(A)_3 \end{bmatrix} P^+
\]

for all \( A \in \mathcal{S} \mathcal{T}_4(F) \), where \( P e_1, P e_2 \in \mathcal{U}_{p,m} \) for some integer \( 1 \leq p \leq \frac{m+1}{2} \), \( \phi(A)_i \) is the \( i \)-th component of \( \phi(A) \in F^3 \), and \( \phi(A) \neq 0 \) for every nonzero bounded rank-two matrix \( A \in \mathcal{S} \mathcal{T}_3(F) \).
Bounded rank two linear preservers

**Theorem 3**

(iv)(b) There exist scalars $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{F}$ with $\lambda_3 \neq 0$ such that either

$$
\psi(A) = P \begin{bmatrix}
\eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} & \eta_1 a_{12} + \lambda_1 a_{qq} \\
\lambda_3 a_{qq} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\
0 & 0 & a_{pp}
\end{bmatrix} \begin{bmatrix}
\eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\
\lambda_3 a_{qq} & \eta_2 a_{12} + a_{13} + \lambda_2 a_{qq} \\
0 & 0 & a_{pp}
\end{bmatrix} P^+
$$

for all $A = (a_{ij}) \in ST_3(\mathbb{F})$, where $\eta_1, \eta_2 \in \mathbb{F}$ are nonzero and $\{p, q\} = \{1, 2\}$; or

$$
\psi(A) = P \begin{bmatrix}
\eta_1 a_{1t} + \lambda_1 a_{qq} \\
a_{1s} + \lambda_2 a_{qq} & \eta_1 a_{1t} + \lambda_1 a_{qq} \\
\lambda_3 a_{qq} & \eta_1 a_{1t} + \lambda_1 a_{qq} \\
0 & 0 & a_{pp}
\end{bmatrix} \begin{bmatrix}
\eta_1 a_{1t} + \lambda_1 a_{qq} \\
a_{1s} + \lambda_2 a_{qq} & \eta_1 a_{1t} + \lambda_1 a_{qq} \\
\lambda_3 a_{qq} & \eta_1 a_{1t} + \lambda_1 a_{qq} \\
0 & 0 & a_{pp}
\end{bmatrix} P^+
$$

for all $A = (a_{ij}) \in ST_3(\mathbb{F})$, where $\eta \in \mathbb{F}$ is nonzero and $\{p, q\} = \{s, t\} = \{1, 2\}$. Here, $P \in M_{m,3}(\mathbb{F})$ is a full rank matrix such that $Pe_1, Pe_2 \in U_{p,m}$ with $1 \leq p \leq \frac{m+1}{2}$ and $Pe_3 \in U_{q,m}$ with $1 \leq q \leq m+1 - p$. In particular, $P \in T_3(\mathbb{F})$ when $m = 3$. 

W. L. Chooi | Linear spaces and preservers of bounded rank two
Why not rank one linear preservers on per-symmetric triangular matrices?

Let \( \mathbb{F} \) be a field and \( m, n \) be integers \( \geq 2 \). Let \( p := \left\lfloor \frac{n+1}{2} \right\rfloor \), where \( \lfloor \cdot \rfloor \) is the floor function. Let \( \psi : ST_n(\mathbb{F}) \to SM_m(\mathbb{F}) \) be the linear mapping defined by

\[
\psi(A) = \lambda P \begin{bmatrix} \phi(A_1) & \varphi(A_2) \\ 0 & \phi(A_1)^+ \end{bmatrix} P^+ \quad \text{for} \quad A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_1^+ \end{bmatrix} \in ST_n(\mathbb{F})
\]

with \( A_1 \in \mathcal{I}_{p,n-p}(\mathbb{F}) \) and \( A_2 \in SM_p(\mathbb{F}) \), where \( \lambda \in \mathbb{F} \backslash \{0\} \), \( P \in \mathcal{M}_{m,n}(\mathbb{F}) \) is of full rank, and \( \phi : \mathcal{I}_{p,n-p}(\mathbb{F}) \to \mathcal{M}_{p,n-p}(\mathbb{F}) \) and \( \varphi : SM_p(\mathbb{F}) \to SM_p(\mathbb{F}) \) are linear mappings. Here \( \mathcal{I}_{p,n-p}(\mathbb{F}) = \mathcal{I}_p(\mathbb{F}) \) when \( n-p = p \), and

\[
\mathcal{I}_{p,n-p}(\mathbb{F}) = \left\{ \begin{bmatrix} T \\ 0 \end{bmatrix} \in \mathcal{M}_{p,n-p}(\mathbb{F}) \mid T \in \mathcal{I}_{p-1}(\mathbb{F}) \right\} \quad \text{when} \quad n-p = p-1.
\]
It can be verified that

1. $\psi$ is a rank one linear preserver whenever $\varphi$ is a rank one linear preserver on $\mathcal{SM}_p(\mathbb{F})$, and
2. $\psi$ is rank one non-increasing whenever $\varphi$ is a rank one non-increasing linear mapping on $\mathcal{SM}_p(\mathbb{F})$.

By the structural results of rank-one linear preservers and rank one non-increasing linear mappings on symmetric matrices, the structure of $\psi$ can be established immediately.
Thank You