9. Title Classical adjoint-commuting mappings on Hermitian and symmetric matrices

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Abstract

Let m and n be integers with $m, n \geqslant 3$, and *let* $\mathbb{F}$ *and* $\mathbb{K}$ *be fields which possess* involutions $^-$ of $\mathbb{F}$ and $\wedge$ of $\mathbb{K}$, respectively. Let $\mathcal{H}_n(\mathbb{F})$ be the $\overline{\mathbb{F}}$-linear space $n \times n$ Hermitian matrices over $\mathbb{F}$. In this note, we address the general form of mappings ψ satisfy one of the following conditions:

1. $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$, *with either* $\left|\widehat{\mathbb{K}}\right| = 2$, *or* $|\overline{\mathbb{F}}|, \left|\widehat{\mathbb{K}}\right| > 3$ *and* $\mathbb{F}$ *and* $\mathbb{K}$ *do not* have characteristic 2 when $^-$ and $\wedge$ are the identity maps, and ψ is surjective satisfying $\psi(\mathrm{adj}(A - B)) = \mathrm{adj}(\psi(A) - \psi(B))$ for every $A, B \in \mathcal{H}_n(\mathbb{F})$.
2. $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{F})$, with either $|\overline{\mathbb{F}}| = 2$ or $|\overline{\mathbb{F}}| > n + 1$, and $\psi(\mathrm{adj}(A + \alpha B)) = \mathrm{adj}(\psi(A) + \alpha\psi(B))$ for every $A, B \in \mathcal{H}_n(\mathbb{F})$ and $\alpha \in \overline{\mathbb{F}}$ with $\psi(I_n) \neq 0$.
3. $\psi : \mathcal{H}_n(\mathbb{F}) \to \mathcal{H}_m(\mathbb{K})$ is additive with $\mathrm{adj}\psi(A) = \psi(\mathrm{adj} A)$ for every $A, B \in \mathcal{H}_n(\mathbb{F})$.

Here, $\overline{\mathbb{F}} := \{a \in \mathbb{F} : \overline{a} = a\}$ and $\widehat{\mathbb{K}} := \{a \in \mathbb{K} : \widehat{a} = a\}$ are the fixed fields of the involutions $^-$ of $\mathbb{F}$ and $\wedge$ of $\mathbb{K}$, respectively, $\mathrm{adj} A$ denotes the classical adjoint of the matrix A, and I_n is the identity matrix of order n.

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10. Title On the Drazin inverse of a class of operators

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Abstract

In this speech, we investigate the formulae for the Drazin inverse of a kind of operators under some conditions. Some equivalent conditions for the Drazin invertibility are established.

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11. Title Maximum norms of commutators of positive contractive operators

Classical adjoint commuting mappings on Hermitian and symmetric matrices

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Introduction

Let A be a square matrix. The classical adjoint of A, denoted adj$A$, is defined by the transposed matrix of cofactors of A.

A mapping ψ between matrix spaces is said to be **classical adjoint commuting** if it satisfies

$$\psi \circ \mathrm{adj} = \mathrm{adj} \circ \psi\,.$$

The structure of classical adjoint commuting linear mappings was first examined by **Sinkhorn** in 1982. He studied such problem on $n \times n$ complex matrices by making use of continuity argument and the Frobenius's classical theorem concerning determinant preservers.

Introduction

He showed, for $n \geqslant 3$, that such mapping is of the *standard* form

$$A \mapsto \lambda PAP^{-1} \quad \text{or} \quad A \mapsto \lambda PA^{t}P^{-1}.$$

Here, P is an invertible complex matrix and λ is a complex scalar with $\lambda^{n-2} = 1$.

Introduction

Since then classical adjoint commuting linear mappings as well as additive mappings on various matrix spaces have been studied. For example

- G.H. Chan, M.H. Lim, K.K. Tan, *Linear preservers on matrices*, Linear Algebra Appl. **93** (1987) 67-80.
- W.L. Chooi, M.H. Lim, *Linear preservers on triangular matrices*, Linear Algebra Appl. **269** (1998) 241-255.
- X.M. Tang, *Additive rank-*1 *preservers between Hermitian matrix spaces and applications*, Linear Algebra Appl. **395** (2005) 333-342.
- X.M. Tang, X. Zhang, *Additive adjoint preservers between matrix spaces*, Linear and Multilinear Algebra. **54(4)** (2006) 285-300.
- W.L. Chooi, *Rank-one increasing additive maps between block triangular matrix algebras*, Linear and Multilinear Algebra. doi.10.1080/03081080902943628.

Motivation

In 2001 at ILAS 2001 Conference, inspired by Hua's work in the Geometry of Matrices, **Peter Šemrl** gave a connection of the study of Classical Preserver Problems and the Geometry of Matrices, and highlighted the study of

Nonlinear Preserver Problems

where classical preserver problems will be studied by dropping algebraic assumptions such as: linearity, additivity or multiplicativity, and replacing with a weaker assumption.

It is surprising that in some cases of preservers, nice structural results could still be obtained without any algebraic assumption imposed on them.

Motivation

Dolinar and **Šemrl** improved the classical result of **Frobenius** concerning determinant preservers ψ by removing the linearity in the following paper:

- G. Dolinar, P. Šemrl, *Determinant preserving maps on matrix algebras*, Linear Algebra Appl. **348** (2002) 189-192.

In this paper, they studied surjections ψ on complex matrices satisfying a single weaker assumption:

$$\det(\psi(A) + \alpha\psi(B)) = \det(A + \alpha B)$$

for every complex matrices A, B and complex scalar α.

Later on, their work has been improved and followed by V. Tan, F. Wang and C.G. Cao.

Motivation

In this talk, motivated by the works of Sinkhorn and Šemrl, we continue the study of classical adjoint commuting mappings on Hermitian and symmetric matrices.

Let $\mathbb{F}$ be a field which possesses an involution $-$, and $\overline{\mathbb{F}}$ denote the fixed field of $-$. Let $\mathcal{H}_n$ be the space of $n \times n$ Hermitian matrices on $-$ over $\mathbb{F}$. Clearly, if the involution $-$ is identity, then $\mathcal{H}_n = \mathcal{S}_n$ the linear space of $n \times n$ symmetric matrices over $\mathbb{F}$.

We consider mappings $\psi : \mathcal{H}_n \to \mathcal{H}_m$ satisfying ONE of the following conditions: for every $A, B \in \mathcal{H}_n$ and $\alpha \in \overline{\mathbb{F}}$,

$$\text{(AC-1)} \quad \psi(\operatorname{adj}(A + \alpha B)) = \operatorname{adj}(\psi(A) + \alpha\psi(B)).$$

$$\text{(AC-2)} \quad \psi(\operatorname{adj}(A - B)) = \operatorname{adj}(\psi(A) - \psi(B)).$$

Motivation

It is clear that if ψ satisfies (AC-1) or (AC-2), then

- $\psi(0) = 0$,
- ψ is a classical adjoint commuting mapping.

Since $\operatorname{adj} I_n = I_n$, we have either $\psi(I_n) = 0$ or $\operatorname{rank} \psi(I_n) = m$.
If $\psi(I_n) = 0$, then it can be shown that ψ sends rank one matrices to zero. Furthermore, if ψ is additive, then $\psi = 0$.

However, under the current setting, it is NOT enough to guarantee $\psi = 0$.

Indeed, beside the zero mapping, we have examples of nonzero classical adjoint commuting mappings that send rank one matrices to zero.

Examples

Example 1

Let m and n be integers with $m, n \geqslant 3$. Let $f : \overline{\mathbb{F}} \to \overline{\mathbb{F}}$ be a nonzero function and let $\psi_1 : \mathcal{H}_n \to \mathcal{H}_m$ be the mapping defined by

$$\psi_1(A) = \begin{cases} f(a_{11})E_{11} & \text{if } A = (a_{ij}) \text{ is of rank } k \text{ with } 1 < k < n \\ 0 & \text{otherwise.} \end{cases}$$

Here, E_{ij} stands for the square matrix unit whose (i, j)-th entry is one and the others are zero.

Examples

Example 2

Let m and n be integers with $m \geqslant n+2$, and let $\phi : \mathcal{H}_n \to \mathcal{H}_n$ be a nonzero mapping. Let $\psi_2 : \mathcal{H}_n \to \mathcal{H}_m$ be the mapping defined by

$$\psi_2(A) = \begin{cases} 0 & \text{if rank } A = 0,\ 1, \text{ or } A \in Adj(GL_n) \\ \phi(A) \oplus 0_{m-n} & \text{otherwise} \end{cases}$$

where $Adj(GL_n) := \{\operatorname{adj} X : X \in \mathcal{H}_n \text{ is invertible}\}$.

Preliminaries

Proposition 1

Let m, n be integers with $m, n \geqslant 3$. If $\psi : \mathcal{H}_n \to \mathcal{H}_m$ is a mapping satisfying (AC-2). Then the following are equivalent

- ψ *is injective.*
- $\operatorname{Ker} \psi = \{0\}$.
- $\operatorname{rank} A = n$ *if and only if* $\operatorname{rank} \psi(A) = m$.

Preliminaries

Proposition 2

Let m, n be integers with $m, n \geqslant 3$. If $\psi : \mathcal{H}_n \to \mathcal{H}_m$ is a mapping satisfying (AC-2). Then the following are equivalent

- $\psi(I_n) = 0$.
- $\psi(A) = 0$ *for all rank one matrices* $A \in \mathcal{H}_n$.
- $\operatorname{rank} \psi(A) \leqslant m - 2$ *for all* $A \in \mathcal{H}_n$.
- $\psi(A) = 0$ *for all* $A \in Adj(GL_n)$.

Preliminaries

Proposition 3

Let m, n be integers with $m, n \geqslant 3$. If $\psi : \mathcal{H}_n \to \mathcal{H}_m$ is a mapping satisfying (AC-2). Then the following are equivalent

- $\psi(I_n) \neq 0$.
- ψ *is injective.*
- $\operatorname{rank}(\psi(A) - \psi(B)) = m$ *if and only if* $\operatorname{rank}(A - B) = n$.

Diameter Preserving Maps or Bounded Distance Preservers

1. H. Havlicek, P. Šemrl, From geometry to invertibility preservers, Studia Math. **174** (2006) 99-109.
2. M.H. Lim, J.J.H. Tan, Preservers of matrix pairs with bounded distance, Linear Algebra Appl. **422** (2007) 517-525.

Main Results

Theorem 1

Let m, n be integers $\geqslant 3$. Let $\mathbb{F}$ be a field which possesses an involution $-$ such that either $\left|\overline{\mathbb{F}}\right| = 2$, or $\left|\overline{\mathbb{F}}\right| > 3$ and $\operatorname{char} \mathbb{F} \neq 2$ when $-$ is identity. Then $\psi : \mathcal{H}_n \to \mathcal{H}_m$ is a surjective mapping satisfying (AC-2) if and only if $m = n$ and

$$\psi(A) = \varsigma P A^{\sigma} \overline{P}^t \quad \text{for all } A \in \mathcal{H}_n.$$

Here, σ is a field isomorphism with $\sigma(\overline{a}) = \overline{\sigma(a)}$ for every $a \in \mathbb{F}$, A^{σ} is the matrix obtained from A applying σ entrywise, P is an invertible n-square matrix with $\overline{P}^t P = \lambda I_n$ and $\lambda, \varsigma \in \overline{\mathbb{F}}$ with $(\varsigma\lambda)^{n-2} = 1$.

Idea of Proof of Theorem 1

- If ψ is a surjection satisfying (AC-2), then $\psi(I_n) \neq 0$. Thus ψ is injective.
- By the injectivity, we show that ψ is a bounded distance preserving bijections.
- It follows from the result of the following papers:
  1. M.H. Lim, J.J.H. Tan, Preservers of matrix pairs with bounded distance, Linear Algebra Appl. **422** (2007) 517-525,
  2. W.L. Huang, H. Havlicek, Diameter preserving surjections in the geometry of matrices, Linear Algebra Appl. **429** (2008) 376-386,

  we conclude that ψ preserves adjacency in both directions.
- Hence, by the fundamental theorem of geometry of Hermitian matrices
  1. Z.X. Wan, Geometry of matrices, World Scientific, Singapore, 1996

  Theorem 1 is proved.

Main Results

Theorem 2

Let m, n be integers $\geqslant 3$. Let $\mathbb{F}$ be a field which possesses an involution $-$ such that either $|\overline{\mathbb{F}}| = 2$ or $|\overline{\mathbb{F}}| > n+1$. Then $\psi : \mathcal{H}_n \to \mathcal{H}_m$ is a mapping satisfying (AC-1) with $\psi(I_n) \neq 0$ if and only if $m = n$ and

$$\psi(A) = \varsigma P A \overline{P}^t \quad \text{for all } A \in \mathcal{H}_n.$$

Here, P is an invertible n-square matrix with $\overline{P}^t P = \lambda I_n$ and $\lambda, \varsigma \in \overline{\mathbb{F}}$ with $(\varsigma\lambda)^{n-2} = 1$.

Idea of Proof of Theorem 2

- If ψ satisfies condition (AC-1) with $\psi(I_n) \neq 0$, then ψ is injective.
- By the injectivity, we get

$$\operatorname{rank}(\psi(A) + \alpha\psi(B)) = m \quad \Leftrightarrow \quad \operatorname{rank}(A + \alpha B) = n.$$

  for any $A, B \in \mathcal{H}_n$ and $\alpha \in \overline{\mathbb{F}}$.
- Then, it follows from the identity

$$\psi(A + \alpha B) = \left(\frac{\det \psi(A + \alpha B)}{\det(\psi(A) + \alpha\psi(B))} \right) (\psi(A) + \alpha\psi(B))$$

  for every $A, B \in \mathcal{H}_n$ and $\alpha \in \overline{\mathbb{F}}$ with $A + \alpha B$ being of rank n, we show that ψ is additive (or more precisely, $\overline{\mathbb{F}}$-linear).

Main Results

Theorem 3

Let m, n be integers $\geqslant 3$. Let $\mathbb{F}$ be a field which possesses an involution $^-$. Then $\psi : \mathcal{H}_n \to \mathcal{H}_m$ is an adjoint-commuting additive mapping if and only if either $\psi = 0$, or $m = n$ and

$$\psi(A) = \varsigma P A^{\sigma} \overline{P}^t \quad \text{for all } A \in \mathcal{H}_n.$$

Here, σ is a field isomorphism with $\sigma(\overline{a}) = \overline{\sigma(a)}$ for every $a \in \mathbb{F}$, A^{σ} is the matrix obtained from A applying σ entrywise, P is an invertible n-square matrix with $\overline{P}^t P = \lambda I_n$ and $\lambda, \varsigma \in \overline{\mathbb{F}}$ with $(\varsigma\lambda)^{n-2} = 1$.