Warped Product Pseudo-Slant Submanifolds of Nearly Kaehler Manifolds

By:

Siraj Uddin

(Paper presented at the ***Differential Geometry and Its Applications Conference*** held on 27-31 August 2010 in Brno, Czech Republic 2010)

Faculty of Science
Masaryk University
Brno, Czech Republic

DGA Differential Geometry and its Applications
Organizing Committee

CONFIRMATION

This is to certify that **Siraj Uddin** (University of Malaya, Kuala Lumpur, Malaysia) participated in the conference *DGA 2010 — the Differential Geometry and its Applications* from August 27[th] till August 31[th] and presented the talk *Warped product pseudo-slant submanifolds of nearly Kaehler manifolds* there.

Brno, August 31[th], 2010

MASARYK UNIVERSITY
FACULTY OF SCIENCE
DEPARTMENT OF MATHEMATICS AND STATISTICS
611 37 Brno, Kotlářská 2

Warped product pseudo-slant submanifolds of nearly Kaehler manifolds

Siraj Uddin (Kuala Lumpur)

Abstract

In this paper, we study warped product pseudo-slant submanifolds of nearly Kaehler manifolds. We prove the non-existence results on warped product submanifolds of nearly Kaehler manifolds.

Key words: Warped product, slant submanifold, pseudo-slant submanifold, nearly Kaehler manifold.

1 Introduction

Slant submanifolds of an almost Hermitian manifold were defined by B.Y. Chen [3] as a natural generalization of both holomorphic and totally real submanifolds. Since then many researchers have studied these submanifolds in complex as well as contact setting [2, 8]. The notion of semi-slant submanifolds of an almost Hermitian manifold was introduced by N. Papaghiuc [9], and is in fact a generalization of CR-submanifolds. Pseudo-slant submanifolds were introduced by A. Carriazo [2] as a special case of bi-slant submanifolds.

Recently, B. Sahin [10] introduced the notion of warped product hemi-slant (pseudo-slant) submanifolds of Kaehler manifolds. He showed that there exist no warped product hemi-slant submanifolds in the form $M_\perp \times {}_f M_\theta$. He considered warped product hemi-slant submanifolds in the form $M_\theta \times {}_f M_\perp$ where $M_\perp$ is a totally real submanifold and M_θ is a proper slant submanifold of a Kaehler manifold and gave some examples for their existence. In this paper we extend the result of B. Sahin in this more general setting. We prove that there do not exist warped product submanifolds of the types $N_\perp \times {}_f N_\theta$ and $N_\theta \times {}_f N_\perp$ in a nearly Kaehler manifold $\bar{M}$, where $N_\perp$ is a totally real submanifold and N_θ is a proper slant submanifold of $\bar{M}$.

2 Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g such that

$$(a)\ \ J^2 = -I, \qquad (b)\ \ g(JX, JY) = g(X, Y) \tag{2.1}$$

for all vector fields X, Y on $\bar{M}$.

Further let $T\bar{M}$ denote the tangent bundle of $\bar{M}$ and $\bar{\nabla}$, the covariant differential operator on $\bar{M}$ with respect to g. If the almost complex structure J satisfies

$$(\bar{\nabla}_X J)X = 0 \tag{2.2}$$

2000 AMS Mathematics Subject Classification: 53C40, 53C42, 53C15.

for any $X \in T\bar{M}$, then the manifold $\bar{M}$ is called a *nearly Kaehler manifold.* Equation (2.2) is equivalent to $(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0$. Obviously, every Kaehler manifold is nearly Kaehler manifold.

For a submanifold M of a Riemannian manifold $\bar{M}$, the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.3}$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N \tag{2.4}$$

for all $X, Y \in TM$, where ∇ is the induced Riemannian connection on M, N is a vector field normal to $\bar{M}$, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle $T^{\perp}M$ and A_N is the shape operator of the second fundamental form. They are related as [11]

$$g(A_N X, Y) = g(h(X,Y), N) \tag{2.5}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the metric induced on M. The mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_{\alpha=1}^{n} h(e_i, e_i) \tag{2.6}$$

where n is the dimension of M and $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of vector fields on M.

A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *totally umbilical submanifold* if the second fundamental form satisfies

$$h(X,Y) = g(X,Y)H \tag{2.7}$$

for all $X, Y \in TM$. The submanifold M is *totally geodesic* if $h(X,Y) = 0$, for all $X, Y \in TM$ and minimal if $H = 0$.

For any $X \in TM$ and $N \in T^{\perp}M$, the transformations JX and JN are decomposed into tangential and normal parts respectively as

$$JX = TX + FX \tag{2.8}$$

$$JN = BN + CN. \tag{2.9}$$

Now, denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\bar{\nabla}_X J)Y$, i.e.,

$$(\bar{\nabla}_X J)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \tag{2.10}$$

for all $X, Y \in TM$. Making use of equations (2.8), (2.9) and the Gauss and Weingarten formulae, the following equations may easily be obtained

$$\mathcal{P}_X Y = (\bar{\nabla}_X T)Y - A_{FY}X - Bh(X,Y) \tag{2.11}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X F)Y + h(X,TY) - Ch(X,Y) \tag{2.12}$$

Similarly, for any $N \in T^{\perp}M$, denoting tangential and normal parts of $(\bar{\nabla}_X J)N$ by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$ respectively, we obtain

$$\mathcal{P}_X N = (\bar{\nabla}_X B)N + TA_N X - A_{CN}X \tag{2.13}$$

$$\mathcal{Q}_X N = (\bar{\nabla}_X C)N + h(BN, X) + FA_N X \tag{2.14}$$

where the covariant derivative of T, F, B and C are defined by

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \tag{2.15}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y \tag{2.16}$$

$$(\bar{\nabla}_X B)N = \nabla_X BN - B\nabla^{\perp}_X N \tag{2.17}$$

$$(\bar{\nabla}_X C)N = \nabla^{\perp}_X CN - C\nabla^{\perp}_X N \tag{2.18}$$

for all $X, Y \in TM$ and $N \in T^{\perp}M$.

It is straightforward to verify the following properties of $\mathcal{P}$ and $\mathcal{Q}$, which we enlist here for later use

(p_1) (i) $\mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W,$ (ii) $\mathcal{Q}_{X+Y}W = \mathcal{Q}_X W + \mathcal{Q}_Y W,$

(p_2) (i) $\mathcal{P}_X(Y+W) = \mathcal{P}_X Y + \mathcal{P}_X W,$ (ii) $\mathcal{Q}_X(Y+W) = \mathcal{Q}_X Y + \mathcal{Q}_X W,$

(p_3) (i) $g(\mathcal{P}_X Y,\ W) = -g(Y, \mathcal{P}_X W),$ (ii) $g(\mathcal{Q}_X Y,\ N) = -g(Y, \mathcal{P}_X N),$

(p_4) $\mathcal{P}_X JY + \mathcal{Q}_X JY = -J(\mathcal{P}_X Y + \mathcal{Q}_X Y)$

for all $X, Y, W \in TM$ and $N \in T^{\perp}M$.

On a submanifold M of a nearly Kaehler manifold, by equations (2.2) and (2.10), we have

$$(a)\ \mathcal{P}_X Y + \mathcal{P}_Y X = 0,\quad (b)\ \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0 \tag{2.19}$$

for any $X,\ Y \in TM$.

The submanifold M is said to be *holomorphic* if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand M is said to be *anti-invariant* if T is identically zero, that is $\phi X \in T^{\perp}M$, for any $X \in TM$.

A distribution D on a submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *slant distribution* if for each $X \in D_x$, the angle θ between JX and D_x is constant i.e., independent of $x \in M$ and $X \in D_x$. In this case, a submanifold M of $\bar{M}$ is said to be a *slant submanifold* if the tangent bundle TM of M is slant.

Moreover, for a slant distribution D, we have

$$T^2 X = -\cos^2\theta X \tag{2.20}$$

for any $X \in D$. Following relations are straightforward consequence of equation (2.20),

$$g(TX, TY) = \cos^2\theta g(X, Y) \tag{2.21}$$

$$g(FX, FY) = \sin^2\theta g(X, Y) \tag{2.22}$$

for all $X, Y \in D$.

A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a *pseudo-slant submanifold* if there exist two orthogonal complementary distributions D_1 and D_2 satisfying:

(i) $TM = D_1 \oplus D_2$

(ii) D_1 is a slant distribution with slant angle $\theta \neq \pi/2$

(iii) D_2 is totally real i.e., $JD_2 \subseteq T^{\perp}M$.

A pseudo-slant submanifold M of an almost Hermitian manifold $\bar{M}$ is *mixed geodesic* if

$$h(X, Z) = 0 \tag{2.23}$$

for any $X \in D_1$ and $Z \in D_2$.

If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of pseudo-slant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as follows

$$T^{\perp}M = \mu \oplus FD_1 \oplus FD_2. \tag{2.24}$$

3 Warped product pseudo-slant submanifolds

In 1969 Bishop and O'Neill [1] introduced the notion of warped product manifolds. These manifolds are natural generalizations of Riemannian product manifolds. They defined these manifolds as: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds and f, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the Riemannian manifold $N_1 \times {}_f N_2 = (N_1 \times N_2, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

A warped product manifold $N_1 \times {}_f N_2$ is said to be *trivial* if the warping function f is constant. We recall the following general formula on a warped product [1].

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \tag{3.2}$$

where X is tangent to N_1 and Z is tangent to N_2.

Let $M = N_1 \times {}_f N_2$ be a warped product manifold. This means that N_1 is totally geodesic and N_2 is a totally umbilical submanifold of M, respectively [1].

Throughout this section, we consider warped product pseudo-slant submanifolds which are either in the form $N_{\perp} \times {}_f N_{\theta}$ or $N_{\theta} \times {}_f N_{\perp}$ in a nearly Kaehler manifold $\bar{M}$, where N_{θ} and $N_{\perp}$ are proper slant and totally real submanifolds of a nearly Kaehler manifold $\bar{M}$, respectively. In the following theorem we consider the warped product pseudo-slant submanifolds in the form $M = N_{\perp} \times {}_f N_{\theta}$ of a nearly Kaehler manifold $\bar{M}$.

Theorem 3.1. *Let $\bar{M}$ be a nearly Kaehler manifold. Then the warped product submanifold $M = N_{\perp} \times {}_f N_{\theta}$ is a Riemannian product of $N_{\perp}$ and N_{θ} if and only if $\mathcal{P}_X TX$ lies in TN_{θ}, for any $X \in TN_{\theta}$, where N_{θ} is a proper slant submanifold and $N_{\perp}$ is a totally real submanifold of $\bar{M}$.*

Proof. Let $M = N_{\perp} \times {}_f N_{\theta}$ be a warped product pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$. For any $X \in TN_{\theta}$ and $W \in TN_{\perp}$, we have

$$g(h(TX, W), FX) = g(\bar{\nabla}_W TX, FX) = -g(TX, \bar{\nabla}_W FX).$$

Using (2.8), we derive

$$g(h(TX, W), FX) = g(TX, \bar{\nabla}_W TX) - g(TX, \bar{\nabla}_W JX).$$

Then from (2.3) and the covariant derivative property of J, we obtain

$$g(h(TX,W),FX) = g(TX,\nabla_W TX) - g(TX,(\bar{\nabla}_W J)X) - g(TX,J\bar{\nabla}_W X).$$

Thus, using (2.1), (2.10) and (3.2) we get

$$g(h(TX,W),FX) = (W\ln f)g(TX,TX) - g(TX,\mathcal{P}_W X) + g(JTX,\bar{\nabla}_W X).$$

Using (2.3), (2.8), (2.19) (a) and (2.21), we obtain

$$g(h(TX,W),FX) = (W\ln f)\cos^2\theta\|X\|^2 + g(TX,\mathcal{P}_X W)$$
$$+ g(T^2X,\nabla_W X) + g(h(X,W),FTX).$$

Thus by property p_3 (i), (2.20) and (3.2), we derive

$$g(h(TX,W),FX) = (W\ln f)\cos^2\theta\|X\|^2 - g(\mathcal{P}_X TX,W)$$
$$- (W\ln f)\cos^2\theta\|X\|^2 + g(h(X,W),FTX).$$

Hence the above equation takes the form

$$g(\mathcal{P}_X TX,W) = g(h(X,W),FTX) - g(h(TX,W),FX). \qquad (3.3)$$

On the other hand for any $X \in TN_\theta$ and $W \in TN_\perp$, we have

$$g(h(X,TX),JW) = g(\bar{\nabla}_{TX}X,JW) = -g(J\bar{\nabla}_{TX}X,W).$$

Using the covariant differentiation formula of J, we get

$$g(h(X,TX),JW) = g((\bar{\nabla}_{TX}J)X,W) - g(\bar{\nabla}_{TX}JX,W).$$

Then by (2.10) and property of $\bar{\nabla}$, we derive

$$g(h(X,TX),JW) = g(\mathcal{P}_{TX}X,W) + g(JX,\bar{\nabla}_{TX}W).$$

Thus from (2.3), (2.8) and (2.19) (a), we obtain

$$g(h(X,TX),JW) = -g(\mathcal{P}_X TX,W) + g(TX,\nabla_{TX}W) + g(h(TX,W),FX).$$

Then from (3.2), the above equation reduces to

$$g(h(X,TX),JW) = -g(\mathcal{P}_X TX,W)$$
$$+ (W\ln f)g(TX,TX) + g(h(TX,W),FX).$$

Hence, using (2.21), we get

$$g(h(X,TX),JW) = -g(\mathcal{P}_X TX,W) + (W\ln f)\cos^2\theta\|X\|^2$$
$$+ g(h(TX,W),FX). \qquad (3.4)$$

By property (p_3) (i), the above equation reduces to

$$g(h(X,TX),JW) = g(TX,\mathcal{P}_X W) + (W\ln f)\cos^2\theta\|X\|^2$$
$$+ g(h(TX,W),FX).$$

Interchanging X by TX and then using (2.20) and (2.21), we obtain

$$-\cos^2\theta g(h(X,TX),JW) = -\cos^2\theta g(X,\mathcal{P}_{TX}W) + (W\ln f)\cos^4\theta g(X,X)$$
$$-\cos^2\theta g(h(X,W),FTX).$$

Again, using first property (p_3) (i) and then (2.19) (a) we arrive at

$$-g(h(X,TX),JW) = -g(\mathcal{P}_X TX,W) + (W\ln f)\cos^2\theta\|X\|^2$$
$$-g(h(X,W),FTX). \quad (3.5)$$

Then from (3.4) and (3.5), we obtain

$$2(W\ln f)\cos^2\theta\|X\|^2 = 2g(\mathcal{P}_X TX,W) + g(h(X,W),FTX)$$
$$-g(h(TX,W),FX). \quad (3.6)$$

Thus, by (3.3) and (3.6), we conclude that

$$(W\ln f)\cos^2\theta\|X\|^2 = \frac{3}{2}g(\mathcal{P}_X TX,W). \quad (3.7)$$

Since N_θ is proper slant, thus we get $(W\ln f)=0$, if and only if $\mathcal{P}_X TX$ lies in TN_θ for all $X\in TN_\theta$ and $W\in TN_\perp$. This shows that f is constant on $N_\perp$. This completes the proof of the theorem. ■

Theorem 3.2. *The warped product submanifold $M = N_\theta \times {}_f N_\perp$ of a nearly Kaehler manifold $\bar{M}$ is simply a Riemannian product of N_θ and $N_\perp$ if and only if*

$$g(h(X,Z),FZ) = g(h(Z,Z),FX), \quad (3.7)$$

for any $X\in TN_\theta$ and $Z\in TN_\perp$, where N_θ is a proper slant submanifold and $N_\perp$ is a totally real submanifold of $\bar{M}$, respectively.

Proof. Let $M = N_\theta\times{}_f N_\perp$ be a warped product submanifold of a nearly Kaehler manifold $\bar{M}$. Then for any $X\in TN_\theta$ and $Z\in TN_\perp$, we have

$$g(h(TX,Z),FZ) = g(\bar{\nabla}_Z TX, JZ).$$

Using (2.1), we get

$$g(h(TX,Z),FZ) = -g(J\bar{\nabla}_Z TX, Z).$$

Thus, on using the covariant differentiation property of J, we obtain

$$g(h(TX,Z),FZ) = g((\bar{\nabla}_Z J)TX,Z) - g(\bar{\nabla}_Z JTX,Z).$$

Then from (2.8) and (2.10), we derive

$$g(h(TX,Z),FZ) = g(\mathcal{P}_Z TX,Z) - g(\bar{\nabla}_Z T^2X,Z) - g(\bar{\nabla}_Z FTX,Z).$$

Now, using (2.4), (p_3) (i) and (2.20) we obtain that

$$g(h(TX,Z),FZ) = -g(\mathcal{P}_Z Z,TX) + \cos^2\theta g(\nabla_Z X,Z) + g(A_{FTX}Z,Z).$$

Since on using (2.2) and (2.10) we have $\mathcal{P}_Z Z = 0$, then from (2.5) and (3.2), we get

$$g(h(TX, Z), FZ) = (X \ln f) \cos^2 \theta \|Z\|^2 + g(h(Z, Z), FTX). \tag{3.8}$$

Interchanging X by TX in (3.8), we obtain

$$\cos^2 \theta g(h(X, Z), FZ) = -(TX \ln f) \cos^2 \theta \|Z\|^2 + \cos^2 \theta g(h(Z, Z), FX).$$

The above equation can be written as

$$(TX \ln f)\|Z\|^2 = g(h(Z, Z), FX) - g(h(X, Z), FZ). \tag{3.9}$$

Thus, $(TX \ln f) = 0$ if and only if $g(h(Z, Z), FX) = g(h(X, Z), FZ)$. This proves the theorem. ■

The following corollaries are the consequences of the above theorem.

Corollary 3.1. *There exists no warped product pseudo-slant submanifold $M = N_\theta \times {}_f N_\perp$ of a nearly Kaehler manifold $\bar{M}$, if the given condition holds*

$$h(TM, D^\perp) \in \mu,$$

where μ is the invariant normal subbundle of TM and $D^\perp$ is a distribution corresponding to the submanifold $N_\perp$.

Proof. The proof follows from (3.9). ■

Corollary 3.2. *There exists no mixed totally geodesic pseudo-slant warped product submanifold $M = N_\theta \times {}_f N_\perp$ of a nearly Kaehler manifold $\bar{M}$ such that $h(Z, Z) \in \mu$ for all $Z \in D^\perp$.*

References

[1] R.L. Bishop and B. O'Neill, *Manifolds of negative curvature,* Trans. Amer. Math. Soc. 145(1969), 1-49.

[2] A. Carriazo, *New developments in slant submanifolds theory,* Narosa Publishing House, New Delhi, India, 2002.

[3] B.Y. Chen, *Slant immersions,* Bull. Austral. Math. Soc. 41(1990), 135-147.

[4] B.Y. Chen, , *Geometry of warped product CR-submanifolds in Kaehler manifolds,* Monatsh. Math. 133(2001), 177-195.

[5] A. Gray, *Nearly Kaehler manifolds,* J. Diff. Geom. 4(1970), 283-309.

[6] V.A. Khan and M.A. Khan, *Pseudo-slant submanifolds of a Sasakian manifold,* Indian J. pure appl. Math. 38(2007), 31-42.

[7] V.A. Khan and K.A. Khan, *Generic warped product submanifolds in nearly Kaehler manifolds,* Beitrge zur Algebra und Geometrie 50(2009), 337-352.

[8] A. Lotta, *Slant submanifolds in contact geometry*, Bull. Math. Soc. Roumanie 39(1996), 183-198.

[9] N. Papaghiuc, *Semi-slant submanifolds of Kaehlerian manifold*, Ann. St. Univ. Iasi 9(1994), 55-61.

[10] B. Sahin, *Warped product submanifolds of Kaehler manifolds with a slant factor*, Ann. Pol. Math. 95(2009), 207-226.

[11] K. Yano, and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, World Scientific Publishing co., Singapore, 1984.

Siraj Uddin:
Institute of Mathematical Sciences,
Faculty of Science,University of Malaya,
50603 Kuala Lumpur, MALAYSIA
E-mail: siraj.ch@gmail.com